Border basis, Hilbert Scheme of points and flat deformations

Mariemi Alonso
(from joint paper with Jerome Brachat and Bernard Mourrain)

Univ. Complutense de Madrid, Spain

ACA2018, Santiago de Compostela
18-22 $^{\text{th}}$ Junio 2018

General Techniques in Computer Algebra

General Techniques in Computer Algebra

- Some methods **are not stable** under perturbation: *Gröbner basis*, *Macaulay H-basis*, *Involutive bases (Janet, Pommaret, etc)*

General Techniques in Computer Algebra

- Some methods **are not stable** under perturbation: *Gröbner basis*, *Macaulay H-basis*, *Involutive bases (Janet, Pommaret, etc)*

DRAWBACKS:

In applications, where the aim is to develop efficient methods which are stable under perturbations. **Starting with a perturbation of the input, do we get nearby output ?**

General Techniques in Computer Algebra

- Some methods **are not stable** under perturbation: *Gröbner basis*, *Macaulay H-basis*, *Involutive bases (Janet, Pommaret, etc)*

DRAWBACKS:

In applications, where the aim is to develop efficient methods which are stable under perturbations. **Starting with a perturbation of the input, do we get nearby output ?**

- Some methods **are stable** under perturbation:

Resultants, Cartan 1945; Kuranishi 1957;

Border basis: Mourrain, Trébuchet: 1999 -2008; and Kehrein, Kreuzer, Robbiano: 2005-2008.

Outline

Outline

Outline

Outline

Outline

0-dim. A-algebras with monomial basis B

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).

0-dim. *A*-algebras with monomial basis *B*

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- Let A be a ring, R^A the ring $A[x_1, \ldots, x_n]$ and let $\mathcal{A}$ be a quotient algebra of $A[x_1, \ldots, x_n]$ such that $\mathcal{A}$ is a free A-module with basis B.

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \dots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- Let A be a ring, R^A the ring $A[x_1, \dots, x_n]$ and let $\mathcal{A}$ be a quotient algebra of $A[x_1, \dots, x_n]$ such that $\mathcal{A}$ is a free A-module with basis B.
- For any $\alpha \in \partial B$, the monomial $\underline{\mathbf{x}}^\alpha$ is a linear combination in A of the monomials of B.

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- Let A be a ring, R^A the ring $A[x_1, \ldots, x_n]$ and let $\mathcal{A}$ be a quotient algebra of $A[x_1, \ldots, x_n]$ such that $\mathcal{A}$ is a free A-module with basis B.
- For any $\alpha \in \partial B$, the monomial $\underline{\mathbf{x}}^\alpha$ is a linear combination in A of the monomials of B. For any $\alpha \in \partial B$, there exists $z_{\alpha,\beta} \in A$ $(\beta \in B)$ s.t.

$$h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}}) := \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta \equiv 0$$

.

0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- Let A be a ring, R^A the ring $A[x_1, \ldots, x_n]$ and let $\mathcal{A}$ be a quotient algebra of $A[x_1, \ldots, x_n]$ such that $\mathcal{A}$ is a free A-module with basis B.
- For any $\alpha \in \partial B$, the monomial $\underline{\mathbf{x}}^\alpha$ is a linear combination in A of the monomials of B. For any $\alpha \in \partial B$, there exists $z_{\alpha,\beta} \in A$ $(\beta \in B)$ s.t.

$$h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}}) := \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta \equiv 0$$

. The $h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}})$ will be called, **the border relations** of $\mathcal{A}$ w.r.t. B.

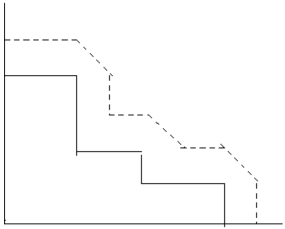
0-dim. A-algebras with monomial basis B

- Let B be set of μ monomials in $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$. We identify B with a set of $\mathbb{N}^n$.
- We assume that B is stable by division, (or connected to 1 (for $\underline{\mathbf{x}}^\alpha \in B$, there is $i : x^\alpha / x_i \in B$).
- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- Let A be a ring, R^A the ring $A[x_1, \ldots, x_n]$ and let $\mathcal{A}$ be a quotient algebra of $A[x_1, \ldots, x_n]$ such that $\mathcal{A}$ is a free A-module with basis B.
- For any $\alpha \in \partial B$, the monomial $\underline{\mathbf{x}}^\alpha$ is a linear combination in A of the monomials of B. For any $\alpha \in \partial B$, there exists $z_{\alpha,\beta} \in A$ ($\beta \in B$) s.t.

$$h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}}) := \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta \equiv 0$$

. The $h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}})$ will be called, **the border relations** of $\mathcal{A}$ w.r.t. B.

- Border relations, are **re-writing rules**



Define a "normal form" , $N^{\mathbf{z}}$

For $\beta \in B$, $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\beta}) = \underline{\mathbf{x}}^{\beta}$,

For $\alpha \in \partial B$. $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\alpha}) = \underline{\mathbf{x}}^{\alpha} - h_{\alpha}^{\mathbf{z}}(\underline{\mathbf{x}}) = \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^{\beta}$

Define a "normal form" , $N^{\mathbf{z}}$

For $\beta \in B$, $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\beta}) = \underline{\mathbf{x}}^{\beta}$,

For $\alpha \in \partial B$. $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\alpha}) = \underline{\mathbf{x}}^{\alpha} - h^{\mathbf{z}}_{\alpha}(\underline{\mathbf{x}}) = \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^{\beta}$

- The tables of multiplication $M^{\mathbf{z}}_{x_i} : \langle B \rangle \to \langle B \rangle$ are constructed using $M^{\mathbf{z}}_{x_i}(\underline{\mathbf{x}}^{\beta}) = N^{\mathbf{z}}(x_i \underline{\mathbf{x}}^{\beta})$ for $\beta \in B$. **These operators of multiplication commute**.

Define a "normal form" , $N^{\mathbf{z}}$

For $\beta \in B$, $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\beta}) = \underline{\mathbf{x}}^{\beta}$,

For $\alpha \in \partial B$. $N^{\mathbf{z}}(\underline{\mathbf{x}}^{\alpha}) = \underline{\mathbf{x}}^{\alpha} - h^{\mathbf{z}}_{\alpha}(\underline{\mathbf{x}}) = \sum_{\beta \in B} z_{\alpha,\beta}\, \underline{\mathbf{x}}^{\beta}$

- The tables of multiplication $M^{\mathbf{z}}_{x_i} : \langle B \rangle \to \langle B \rangle$ are constructed using $M^{\mathbf{z}}_{x_i}(\underline{\mathbf{x}}^{\beta}) = N^{\mathbf{z}}(x_i \underline{\mathbf{x}}^{\beta})$ for $\beta \in B$. **These operators of multiplication commute**.
- Notice that the coefficients of the matrix of $M^{\mathbf{z}}_{x_i}$ in the basis B are linear in the coefficients $\mathbf{z}$'s.

Border equations

- Conversely, if we are interested in characterizing the coefficients $\mathbf{z} := (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ such that the polynomials $(h^{\mathbf{z}}_{\alpha}(\underline{\mathbf{x}}))_{\alpha\in B}$ are the border relations of some free A-algebra $\mathcal{A}^{\mathbf{z}} = A[x_1,\ldots,x_n]/\mathrm{I}$ with basis B.

Border equations

- Conversely, if we are interested in characterizing the coefficients $\mathbf{z} := (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ such that the polynomials $(h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}}))_{\alpha\in B}$ are the border relations of some free A-algebra $\mathcal{A}^{\mathbf{z}} = A[x_1,\ldots,x_n]/\mathrm{I}$ with basis B.Mourrain '99, and Kreuzer-Robbiano'08 proved

Theorem

Let B be a set of μ monomials connected to 1. The polynomials $h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}})$, $\mathbf{z}\in A$, are the border relations of some free quotient algebra $\mathcal{A}$ of $A[x_1,\ldots,x_n]$ of basis B iff

$$M_{x_i}^{\mathbf{z}}\circ M_{x_j}^{\mathbf{z}} - M_{x_j}^{\mathbf{z}}\circ M_{x_i}^{\mathbf{z}} = 0 \quad \text{for} \quad 1\leqslant i<j\leqslant n. \tag{1}$$

$$\mathcal{H}_B := \{\mathbf{z} = (z_{\alpha,\beta})\in\mathbb{K}^{\partial B\times B};\, M_{x_i}^{\mathbf{z}}\circ M_{x_j}^{\mathbf{z}} - M_{x_j}^{\mathbf{z}}\circ M_{x_i}^{\mathbf{z}} = 0_{\ 1\leqslant i<j\leqslant n}\}$$

Perturbing equations

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.
- Let B be a set of monomials that is also a $\mathbb{K}$-basis for $\mathcal{A}^0$.

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.
- Let B be a set of monomials that is also a $\mathbb{K}$-basis for $\mathcal{A}^0$.
- Let us perturb the system $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots$, and let $A = \mathbb{K}[[\varepsilon]]$, $R^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}] = A[\mathbf{x}]$, $\mathcal{A} := R^\varepsilon/\mathbf{I}$ and, $(\mathbf{f}) = \mathbf{I}$ with $\mathbf{I}^0$ describing the initial finite zero-set.

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.
- Let B be a set of monomials that is also a $\mathbb{K}$-basis for $\mathcal{A}^0$.
- Let us perturb the system $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots$, and let $A = \mathbb{K}[[\varepsilon]]$, $R^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}] = A[\mathbf{x}]$, $\mathcal{A} := R^\varepsilon/\mathbf{I}$ and, $(\mathbf{f}) = \mathbf{I}$ with $\mathbf{I}^0$ describing the initial finite zero-set. **Obstructions for flatness of**

$$\mathbb{K}[[\varepsilon]] \to \mathcal{A}$$

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.
- Let B be a set of monomials that is also a $\mathbb{K}$-basis for $\mathcal{A}^0$.
- Let us perturb the system $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots$, and let $A = \mathbb{K}[[\varepsilon]]$, $R^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}] = A[\mathbf{x}]$, $\mathcal{A} := R^\varepsilon/\mathbf{I}$ and, $(\mathbf{f}) = \mathbf{I}$ with $\mathbf{I}^0$ describing the initial finite zero-set. **Obstructions for flatness of**

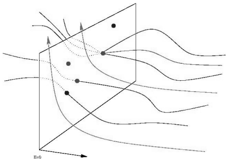
$$\mathbb{K}[[\varepsilon]] \to \mathcal{A}$$

"isolated, embedded points, points going to infinite"

Perturbing equations

- Start with algebraic equations defining a finite set of points $\mathbf{f}^0 \in \mathbb{K}[x_1, \ldots, x_n]^s$, let $\mathbf{I}^0 = (\mathbf{f}^0)$ the 0-dim ideal and $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathbf{I}^0$.
- Let B be a set of monomials that is also a $\mathbb{K}$-basis for $\mathcal{A}^0$.
- Let us perturb the system $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots$, and let $A = \mathbb{K}[[\varepsilon]]$, $R^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}] = A[\mathbf{x}]$, $\mathcal{A} := R^\varepsilon/\mathbf{I}$ and, $(\mathbf{f}) = \mathbf{I}$ with $\mathbf{I}^0$ describing the initial finite zero-set. **Obstructions for flatness of**

$$\mathbb{K}[[\varepsilon]] \to \mathcal{A}$$



Flatness means the monomial basis B is still a basis of $\mathcal{A}$ as $\mathbb{K}[[\varepsilon]]$ module (assumed $\mathcal{A}$ is finite $\mathbb{K}[[\varepsilon]]$– module)

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.
- Consider the multiplicative set $S = \{g(\boldsymbol{x}) \in A[x] : g(\boldsymbol{x}) \bmod \mathfrak{m} = 1\}$

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.
- Consider the multiplicative set $S = \{g(\boldsymbol{x}) \in A[x] : g(\boldsymbol{x}) \bmod \mathfrak{m} = 1\}$
- Let $\mathcal{A}_a := S^{-1}\mathcal{A} = S^{-1}A[\mathbf{x}]/\mathrm{I}$.

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\,\mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.
- Consider the multiplicative set $S = \{g(\boldsymbol{x}) \in A[x] : g(\boldsymbol{x}) \bmod \mathfrak{m} = \ 1\}$
- Let $\mathcal{A}_a := S^{-1}\mathcal{A} = S^{-1}A[\mathbf{x}]/\mathrm{I}$.

The effect of taking the extended ring is to keep only "the points to finite distance"

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.
- Consider the multiplicative set

$S = \{g(\boldsymbol{x}) \in A[x] : g(\boldsymbol{x}) \bmod \mathfrak{m} = 1\}$

- Let $\mathcal{A}_a := S^{-1}\mathcal{A} = S^{-1}A[\mathbf{x}]/\mathrm{I}$.

The effect of taking the extended ring is to keep only "the points to finite distance" The ring $S^{-1}\mathcal{A}$ is a finite A-module.

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon\, \mathbf{f}^1 + \cdots ; \varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.
- Consider the multiplicative set $S = \{g(\boldsymbol{x}) \in A[x] : g(\boldsymbol{x}) \bmod \mathfrak{m} = 1\}$
- Let $\mathcal{A}_a := S^{-1}\mathcal{A} = S^{-1}A[\mathbf{x}]/\mathrm{I}$.

The effect of taking the extended ring is to keep only "the points to finite distance" The ring $S^{-1}\mathcal{A}$ is a finite A-module.

QUESTION:

Conditions for $\mathcal{A} = A[\mathbf{x}]/\mathrm{I}$ (resp. $\mathcal{A}_a = S^{-1}(A[\mathbf{x}]/\mathrm{I})$ to be a flat (hence free) A module?What can we say of a border basis of $\mathcal{A}$ (or $\mathcal{A}_a$), assuming one knows a border basis mod. $\mathfrak{m}$?

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h^0_\beta := x^\beta - \sum z^0_{\alpha\beta}\, x^\alpha \; ; \; z^0_{\alpha\beta} \in \mathbb{K}$$

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h^0_\beta := x^\beta - \sum z^0_{\alpha\beta}\, x^\alpha \; ; \; z^0_{\alpha\beta} \in \mathbb{K}$$

we try to lift them to get border relations in $\mathcal{A}$

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h_\beta^0 := x^\beta - \sum z_{\alpha\beta}^0 \ x^\alpha \ ; \ z_{\alpha\beta}^0 \in \mathbb{K}$$

We can lift h_β^0 to $\mathcal{A}$, getting new elements $\mathbf{f}^\beta$, for $\beta \in \partial B$ s.t. $\mathbf{f}^\beta \in$ I. Then, let

$$\tilde{h_\beta} = x^\beta - \sum z_{\tilde{\alpha}\beta} \mathbf{x}^\alpha$$

for $\beta \in \partial B$ and $\alpha \in B$, where $z_{\tilde{\alpha}\beta}$ are unknowns, that that we try to determine as elements of A s.t. $\quad z_{\alpha\beta} \mod .\mathfrak{m} = z_{\alpha\beta}^0$

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h_\beta^0 := x^\beta - \sum z_{\alpha\beta}^0 \; x^\alpha \; ; \; z_{\alpha\beta}^0 \in \mathbb{K}$$

We can lift h_β^0 to $\mathcal{A}$, getting new elements $\mathbf{f}^\beta$, for $\beta \in \partial B$ s.t. $\mathbf{f}^\beta \in$ I. Then, let

$$\tilde{h_\beta} = x^\beta - \sum z_{\tilde{\alpha\beta}} \mathbf{x}^\alpha$$

for $\beta \in \partial B$ and $\alpha \in B$, where $z_{\tilde{\alpha\beta}}$ are unknowns, that that we try to determine as elements of A s.t. $\quad z_{\alpha\beta} \bmod .\mathfrak{m} = z_{\alpha\beta}^0$

We reduce the generators $\mathbf{f}^\beta$'s with the $\tilde{h_\beta}$'s $\;(\beta \in \partial B$,and we impose the condition that the remainder must be zero .

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h_\beta^0 := x^\beta - \sum z_{\alpha\beta}^0 \ x^\alpha \ ; \ z_{\alpha\beta}^0 \in \mathbb{K}$$

We can lift h_β^0 to $\mathcal{A}$, getting new elements $\mathbf{f}^\beta$, for $\beta \in \partial B$ s.t. $\mathbf{f}^\beta \in$ I. Then, let

$$\tilde{h_\beta} = x^\beta - \sum z_{\tilde{\alpha}\beta} \mathbf{x}^\alpha$$

for $\beta \in \partial B$ and $\alpha \in B$, where $z_{\tilde{\alpha}\beta}$ are unknowns, that that we try to determine as elements of A s.t. $\quad z_{\alpha\beta} \mod . \mathfrak{m} = z_{\alpha\beta}^0$

We reduce the generators $\mathbf{f}^\beta$'s with the $\tilde{h_\beta}$'s $(\beta \in \partial B$,and we impose the condition that the remainder must be zero .

Obtaining a Hensel system with a unique solution $z_{\alpha\beta} \in A$, lifting $z_{\alpha\beta}^0$.

Flatness criterion (conti..)

Starting with border relations for the residual algebra $\mathcal{A}^0$

$$h^0_\beta := x^\beta - \sum z^0_{\alpha\beta}\ x^\alpha \ ;\ z^0_{\alpha\beta} \in \mathbb{K}$$

We can lift h^0_β to $\mathcal{A}$, getting new elements $\mathbf{f}^\beta$, for $\beta \in \partial B$ s.t. $\mathbf{f}^\beta \in \mathrm{I}$. Then, let

$$\tilde{h_\beta} = x^\beta - \sum z_{\tilde{\alpha}\beta}\mathbf{x}^\alpha$$

for $\beta \in \partial B$ and $\alpha \in B$, where $z_{\tilde{\alpha}\beta}$ are unknowns, that that we try to determine as elements of A s.t. $\quad z_{\alpha\beta} \bmod .\mathfrak{m} = z^0_{\alpha\beta}$

We reduce the generators $\mathbf{f}^\beta$'s with the $\tilde{h_\beta}$'s $\ (\beta \in \partial B$,and we impose the condition that the remainder must be zero .

Obtaining a Hensel system with a unique solution $z_{\alpha\beta} \in A$, lifting $z^0_{\alpha\beta}$.Write $h_\beta = x^\beta - \sum z_{\alpha\beta}\mathbf{x}^\alpha$ and set

$$\mathcal{H} := ((h_\beta)_{\beta\in\partial B})S^{-1}A[\mathbf{x}] \subset \mathrm{I}S^{-1}\mathrm{A}[\mathbf{x}]$$

Flatness criterion (conti..)

$$\mathcal{H} := ((h_\beta)_{\beta \in \partial B}) S^{-1} A[\mathbf{x}] \subset \mathrm{I}\mathrm{S}^{-1}\mathrm{A}[\mathbf{x}]$$

WE GET FLATNESS, iff the lifted border relations:

Flatness criterion (conti..)

$$\mathcal{H} := ((h_\beta)_{\beta \in \partial B}) S^{-1} A[\mathbf{x}] \subset \mathrm{I} \mathrm{S}^{-1}\mathrm{A}[\mathbf{x}]$$

WE GET FLATNESS, iff the lifted border relations:

- i) verify the equations of commutativity, in order to be border basis of $A[\mathbf{x}]/(h_{\alpha\beta})$, and
- ii) generate the ideal of the beginning: $\mathrm{I}\ \mathrm{S}^{-1}\mathrm{A}[\mathbf{x}] = \mathcal{H}$ (generators of I reduce to zero $\mathrm{mod.}$ the lifted equations)

Example

We consider the perturbation
$f_1^\varepsilon = x^2 - \varepsilon x$, $f_2^\varepsilon = xy - \varepsilon x$, $f_3^\varepsilon = xy - \varepsilon y$, $f_4^\varepsilon = y^2 - \varepsilon y$, $f_5^\varepsilon = \varepsilon x - \varepsilon^2$, $f_6^\varepsilon = \varepsilon y - \varepsilon^2$.

- We have $\mathrm{I}^0 = (\mathrm{x}^2, \mathrm{xy}, \mathrm{y}^2)$ and $\mathrm{I} = (\mathrm{f}_1^\varepsilon, \dots, \mathrm{f}_6^\varepsilon)$.
- The set $B = \{1, x, y\}$ is a basis of $R/\mathfrak{I}^0$ and the border relations are $h^0_{x^2} = x^2$, $h^0_{xy} = xy$, $h^0_{y^2} = y^2$. As $h^0_{x^2} = f^0_1$, $h^0_{xy} = f^0_2$, $h^0_{y^2} = f^0_4$, these border relations lift in

  - $\tilde{h}^\varepsilon_{x^2} = f_1^\varepsilon = x^2 - \varepsilon x$,
  - $\tilde{h}^\varepsilon_{xy} = f_2^\varepsilon = xy - \varepsilon x$,
  - $\tilde{h}^\varepsilon_{y^2} = f_4^\varepsilon = y^2 - \varepsilon y$.

- After reduction by the formal border relations and resolution of the corresponding (linear) system , we have $h^\varepsilon_m = \tilde{h}^\varepsilon_m$, so that $\mathrm{I}\ \mathrm{S}^{-1}\mathrm{A}[\mathbf{x}] = \mathcal{H}$.
- Only need to chek that the multiplication operators by x and y commute, so that the polynomials $h^\varepsilon_{x^2}, h^\varepsilon_{xy}, h^\varepsilon_{y^2}$ are border relations for B.

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $\mathrm{I}^0 = (\mathrm{y}^2, \mathrm{x}^3 + \mathrm{yx}^2)$ and $\mathrm{I} = (\mathrm{f}_1, \mathrm{f}_2)\mathrm{S}^{-1}\mathbb{K}[[\mathrm{t}]][\mathrm{x}, \mathrm{y}]$ Setting $t = 0$, the system has an isolated zero $(0, 0)$ of multiplicity 6.

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $\mathrm{I}^0 = (\mathrm{y}^2, \mathrm{x}^3 + \mathrm{yx}^2)$ and $\mathrm{I} = (\mathrm{f}_1, \mathrm{f}_2)\mathrm{S}^{-1}\mathbb{K}[[\mathrm{t}]][\mathrm{x}, \mathrm{y}]$ Setting $t = 0$, the system has an isolated zero $(0, 0)$ of multiplicity 6.
- $\mathbf{h^0_{y^2}} = y^2, h^0_{y^2x} = y^2x, h^0_{y^2x^2} = y^2x^2, \mathbf{h^0_{x^3}} = \mathbf{x^3} + \mathbf{x^2y}, \mathbf{h^0_{x^3y}} = \mathrm{red}(yh^0_{x^3}) = \mathbf{x^3y}$ is a border basis for $\mathbb{K}[x, y]/\mathrm{I}^0$, $\partial B = \{1, x, y, x^2, xy, x^2y\}$

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $I^0 = (y^2, x^3 + yx^2)$ and $I = (f_1, f_2)S^{-1}\mathbb{K}[[t]][x,y]$ Setting $t = 0$, the system has an isolated zero $(0,0)$ of multiplicity 6.
- $\mathbf{h^0_{y^2}} = y^2, h^0_{y^2x} = y^2x, h^0_{y^2x^2} = y^2x^2, \mathbf{h^0_{x^3}} = \mathbf{x^3} + \mathbf{x^2y}, \mathbf{h^0_{x^3y}} = \mathrm{red}(yh^0_{x^3}) = \mathbf{x^3y}$ is a border basis for $\mathbb{K}[x,y]/I^0$, $\partial B = \{1, x, y, x^2, xy, x^2y\}$ For t small enough the system has 6 roots be very near to $(0,0)$: a cluster and another more point away it.

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $\mathrm{I}^0 = (\mathrm{y}^2, \mathrm{x}^3 + \mathrm{yx}^2)$ and $\mathrm{I} = (\mathrm{f}_1, \mathrm{f}_2)\mathrm{S}^{-1}\mathbb{K}[[\mathrm{t}]][\mathrm{x}, \mathrm{y}]$
- $\mathbf{h^0_{y^2}} = y^2, h^0_{y^2x} = y^2x, h^0_{y^2x^2} = y^2x^2, \mathbf{h^0_{x^3}} = \mathbf{x^3} + \mathbf{x^2y}, \mathbf{h^0_{x^3y}} =$

$\mathrm{red}(yh^0_{x^3}) = \mathbf{x^3y}$ is a border basis for $\mathbb{K}[x, y]/\mathrm{I}^0$, $\partial B = \{1, x, y, x^2, xy, x^2y\}$ For t small enough the system has 6 roots be very near to $(0, 0)$: a cluster and another more point away it. **Warning, this example doesn't correspond exactly to the last theorem. Here $\mathcal{A}_a$ is not A-flat , a bigger ring $\mathcal{A}_{(\mathfrak{m},x,y)}$ is A-flat. But it allow us to show how to construct efficiently a border basis for the cluster**

Example: *"The importance of being flat" (O. Wild)*

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $I^0 = (y^2, x^3 + yx^2)$ and $I = (f_1, f_2)S^{-1}\mathbb{K}[[t]][x,y]$
- $\mathbf{h^0_{y^2}} = y^2, h^0_{y^2x} = y^2x, h^0_{y^2x^2} = y^2x^2, \mathbf{h^0_{x^3}} = \mathbf{x^3} + \mathbf{x^2y}, \mathbf{h^0_{x^3y}} =$

$\mathrm{red}(yh^0_{x^3}) = \mathbf{x^3y}$ is a border basis for $\mathbb{K}[x,y]/I^0$,
$\partial B = \{1, x, y, x^2, xy, x^2y\}$

- We introduce the $\tilde{h}$'s:

$$\begin{aligned}
\tilde{h_{y^2}} &:= y^2 + a_0 + a_1\, x + a_2\, x^2 + a_3\, y + a_4\, yx + a_5\, yx^2 \\
\tilde{h_{x^3}} &:= x^3 + b_0 + b_1\, x + b_2\, x^2 + b_3\, y + b_4\, yx + b_5\, yx^2 \\
\tilde{h_{x^3y}} &:= x^3y + c_0 + c_1\, x + c_2\, x^2 + c_3\, y + c_4\, xy + c_5\, x^2y
\end{aligned}$$

Example: *"The importance of being flat"* (*O. Wild*)

Example:

$$f_1 := y^2 + tyx^2 + t, \quad f_2 := x^3 + yx^2 - t^2$$

- $I^0 = (y^2, x^3 + yx^2)$ and $I = (f_1, f_2)S^{-1}\mathbb{K}[[t]][x, y]$
- $\mathbf{h^0_{y^2}} = y^2, h^0_{y^2x} = y^2x, h^0_{y^2x^2} = y^2x^2, \mathbf{h^0_{x^3}} = \mathbf{x^3} + \mathbf{x^2y}, \mathbf{h^0_{x^3y}} = \mathrm{red}(yh^0_{x^3}) = \mathbf{x^3y}$ is a border basis for $\mathbb{K}[x, y]/I^0$, $\partial B = \{1, x, y, x^2, xy, x^2y\}$
- We introduce the $\tilde{h}$'s:

$$\begin{aligned} \tilde{h_{y^2}} &:= y^2 + a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 yx + a_5 yx^2 \\ \tilde{h_{x^3}} &:= x^3 + b_0 + b_1 x + b_2 x^2 + b_3 y + b_4 yx + b_5 yx^2 \\ \tilde{h_{x^3y}} &:= x^3y + c_0 + c_1 x + c_2 x^2 + c_3 y + c_4 xy + c_5 x^2y \end{aligned}$$

- reduce f_1, f_2's with $\tilde{h}$'s, obtaining some Hensel equations

$$a_1 = a_2 = a_3 = a_4 = 0, \; b_1 = b_2 = b_3 = b_4 = 0$$

$$a_0 = t, b_0 = -t^2, a_5 = t, b_5 = 1,$$

$$c_2 = \frac{t\left(1+tc_5\right)^2}{-1-3\,tc_5-3\,t^2{c_5}^2-t^3{c_5}^3+t^5}$$

$$c_0 = \frac{t^3c_2}{1+tc_5},\; c_1 = \frac{t^4c_2}{(1+tc_5)^2},\; c_3 = \frac{-t^2}{1+tc_5},\; c_4 = \frac{-t^3}{(1+tc_5)^2}$$

$$-t^6\,c_5^7-6t^5\,c_5^6-15t^4\,c_5^5+(-20t^3+t^8+t^6)\,c_5^4+(-15t^2+4t^5+2t^7)\,c_5^3 \\ +(6t^4-6t)\,c_5^2+(-1-2t^5+4t^3)\,c_5+(t^2+t^9-t^4) \quad =$$

- We approximate till $o(t^{10})$ with Newton method the rational functions, and the "unique" solution of the last equation near zero

$$t^2-t^4+4\,t^5-6\,t^7+16\,t^8+3\,t^9+O\left(t^{10}\right)$$
$$t^2-t^4-2\,t^5+6\,t^7+7\,t^8-3\,t^9+O\left(t^{10}\right)$$
$$t^2-t^4-2\,t^5+6\,t^7+7\,t^8-3\,t^9- \\ 35t^{10}-30\,t^{11}+45\,t^{12}+210\,t^{13}+128\,t^{14}+O\left(t^{15}\right)$$

- Using $c_5 = t^2-t^4-2\,t^5$, or $c_5 = t^2$, leads to the two following (approximating) matrices for the multiplication by y in the basis of monomials under the staircase:

$$aprMy := \begin{bmatrix} 0 & 0 & -t & 0 & -t^3 - t^5 & -t^4 + 2\,t^7 \\ 0 & 0 & 0 & 0 & -t - t^6 & -t^5 \\ 1 & 0 & 0 & 0 & -t^3 + t^6 & t^5 - t^7 \\ 0 & 0 & 0 & 0 & -t^2 + t^5 & -t + t^4 - 2\,t^6 - 3\,t^7 \\ 0 & 1 & 0 & 0 & -t^4 + 2\,t^7 & -t^3 + 2\,t^6 \\ 0 & 0 & -t & 1 & t^3 - t^5 - 2\,t^6 & t^2 - t^4 - 2\,t^5 + 4\,t^7 \end{bmatrix}$$

$$AprMy := \begin{bmatrix} 0 & 0 & -t & 0 & -t^3 & -t^4 \\ 0 & 0 & 0 & 0 & -t & -t^5 \\ 1 & 0 & 0 & 0 & -t^3 & t^5 \\ 0 & 0 & 0 & 0 & -t^2 & -t \\ 0 & 1 & 0 & 0 & -t^4 & -t^3 \\ 0 & 0 & -t & 1 & t^3 & t^2 \end{bmatrix}$$

We may use the characteristic polynomial of one of these matrices. For small values of t, say $\mid t \mid \leq 10^{-2}$, the second one is sufficient:

$$\begin{aligned} Gy \quad = \quad & y^6 + \left(-t^2 + t^4\right) y^5 + \left(t^6 + 3\,t\right) y^4 + \left(t^7 + t^8 + t^{10} - 2\,t^3\right) y^3 \\ & + \left(t^7 + 3\,t^2\right) y^2 + \left(-t^6 + 2\,t^9 - t^4\right) y + \left(t^3 - t^8\right) \end{aligned}$$

Computing with 12 digits we get correct answers up to many digits for the cluster of six roots. The same computation, when using a floating point Gröbner basis computation needs arround 200 digits of precision.

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ = The Hilbert functor of $\mathbb{P}^n$ relative to $\mu \in \mathbb{Z}^+$

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n} = \text{The Hilbert functor of } \mathbb{P}^n \text{ relative to } \mu \in \mathbb{Z}^+$$

$$\mathcal{C} = \text{ Schemes of finite type over } \mathbb{K} \Longrightarrow \text{Sets}$$

$$X \mapsto \{\text{flat families} Z \subset X \times \mathbb{P}^n \text{ with fibers having Hilbert polynomial } \mu\}$$

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ = The Hilbert functor of $\mathbb{P}^n$ relative to $\mu \in \mathbb{Z}^+$

$\mathcal{C} =$ Schemes of finite type over $\mathbb{K} \Longrightarrow$ Sets

$X \mapsto \{$flat families$Z \subset X \times \mathbb{P}^n$ with fibers having Hilbert polynomial $\mu\}$

- If $X = \mathbf{Spec}(A)$, A is a $\mathbb{K}$–algebra of finite type, and the homogeneous ring $S^A = A[x_0, \ldots, x_n]$ ($S^A =: S$ for short)

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{I \subset S^A$ homog. sat. ideal :

S^A_d / I_d is A locally free mod. of rank μ $\forall d >> 0\}$

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ = The Hilbert functor of $\mathbb{P}^n$ relative to $\mu \in \mathbb{Z}^+$

$\mathcal{C} =$ Schemes of finite type over $\mathbb{K} \Longrightarrow$ Sets

$X \mapsto \{$flat families$Z \subset X \times \mathbb{P}^n$ with fibers having Hilbert polynomial $\mu\}$

- If $X = \mathbf{Spec}(A)$, A is a $\mathbb{K}$–algebra of finite type, and the homogeneous ring $S^A = A[x_0, \ldots, x_n]$ ($S^A =: S$ for short)

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{I \subset S^A \text{ homog. sat. ideal :}$$

$$S^A_d / I_d \text{ is } A \text{ locally free mod. of rank } \mu \ \forall d >> 0\}$$

This set has a structure of scheme.

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ = The Hilbert functor of $\mathbb{P}^n$ relative to $\mu \in \mathbb{Z}^+$

$\mathcal{C} =$ Schemes of finite type over $\mathbb{K} \Longrightarrow$ Sets

$X \mapsto \{$flat families$Z \subset X \times \mathbb{P}^n$ with fibers having Hilbert polynomial $\mu\}$

- If $X = \mathbf{Spec}(A)$, A is a $\mathbb{K}$–algebra of finite type, and the homogeneous ring $S^A = A[x_0, \ldots, x_n]$ ($S^A =: S$ for short)

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{I \subset S^A \text{ homog. sat. ideal :}$$

$$S^A_d / I_d \text{ is } A \text{ locally free mod. of rank } \mu \ \forall d >> 0\}$$

- One can cover the functor $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ with an open covering of affine representable subfunctors namely

Construction $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$

$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ = The Hilbert functor of $\mathbb{P}^n$ relative to $\mu \in \mathbb{Z}^+$
$\mathcal{C} =$ Schemes of finite type over $\mathbb{K} \Longrightarrow$ Sets

$X \mapsto \{$flat families$Z \subset X \times \mathbb{P}^n$ with fibers having Hilbert polynomial $\mu\}$

- If $X = \mathbf{Spec}(A)$, A is a $\mathbb{K}$–algebra of finite type, and the homogeneous ring $S^A = A[x_0, \ldots, x_n]$ ($S^A =: S$ for short)

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{I \subset S^A \text{ homog. sat. ideal :}$$

$$S^A_d/I_d \text{ is } A \text{ locally free mod. of rank } \mu \ \forall d >> 0\}$$

- One can cover the functor $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}$ with an open covering of affine representable subfunctors namely $\mathbf{H}^B_u$ (B a set of μ monomials of degree, stable by division and $u \in S_1$; s.t. $\mathbf{H}^B_u$ is represented by $\mathbf{Spec}(\mathbb{K}[(z_{\alpha,\beta})_{\alpha \in \delta B, \beta \in B}]/\mathcal{R})$, where $\mathcal{R}$ is the ideal of *commutating relations.*

Plucker coordinates

We assume A is a local ring.

Plucker coordinates

We assume A is a local ring.

- Let $X = \mathbf{Spec}(A)$, and
  $\mathbf{Gr}^{\mu}_{S_d^*}(X) = \{\Delta : \Delta = S_d/I_d : A \text{ free module of rank } \mu\}$

Plucker coordinates

We assume A is a local ring.

- Let $X = \mathbf{Spec}(A)$, and $\mathbf{Gr}^{\mu}_{S_d^*}(X) = \{\Delta : \Delta = S_d/I_d : A \text{ free module of rank } \mu\}$

$$\mathbf{Gr}^{\mu}_{S_d^*}(X) \hookrightarrow \mathbb{P}(\wedge^{\mu}(S_d^A)^*)$$

:

Plucker coordinates

We assume A is a local ring.

- Let $X = \mathbf{Spec}(A)$, and
  $\mathbf{Gr}^{\mu}_{S_d^*}(X) = \{\Delta : \Delta = S_d/I_d : A \text{ free module of rank } \mu\}$
  $$\mathbf{Gr}^{\mu}_{S_d^*}(X) \hookrightarrow \mathbb{P}(\wedge^{\mu}(S_d^A)^*)$$
- Consider the Plucker coordinates in in $\mathbb{P}(\wedge^{\mu}(S_d^A)^*)$:

Plucker coordinates

We assume A is a local ring.

- Let $X = \mathbf{Spec}(A)$, and
  $\mathbf{Gr}^{\mu}_{S_d^*}(X) = \{\Delta : \Delta = S_d/I_d : A \text{ free module of rank } \mu\}$
  $$\mathbf{Gr}^{\mu}_{S_d^*}(X) \hookrightarrow \mathbb{P}(\wedge^{\mu}(S_d^A)^*)$$
- Consider the Plucker coordinates in in $\mathbb{P}(\wedge^{\mu}(S_d^A)^*)$:
- Let $\Delta = S_d/I_d \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$, and $(\delta_1, \ldots, \delta_{\mu})$ in be any basis of the dual space Δ^* (also a free A-module of rank μ).

Plucker coordinates

We assume A is a local ring.

- Let $X = \mathbf{Spec}(A)$, and $\mathbf{Gr}^{\mu}_{S_d^*}(X) = \{\Delta : \Delta = S_d/I_d : A \text{ free module of rank } \mu\}$
$$\mathbf{Gr}^{\mu}_{S_d^*}(X) \hookrightarrow \mathbb{P}(\wedge^{\mu}(S_d^A)^*)$$
- Consider the Plucker coordinates in in $\mathbb{P}(\wedge^{\mu}(S_d^A)^*)$:
- Let $\Delta = S_d/I_d \in \mathbf{Gr}^{\mu}_{S_d^*}(X)$, and $(\delta_1, \ldots, \delta_\mu)$ in be any basis of the dual space Δ^* (also a free A-module of rank μ).
- Plücker coordinates of Δ as an element of $\mathbb{P}(\wedge^{\mu} S_d^*)$ are given by:
$$\Delta_{\beta_1,\ldots,\beta_\mu} = \begin{vmatrix} \delta_1(\mathbf{x}^{\beta_1}) & \cdots & \delta_1(\mathbf{x}^{\beta_\mu}) \\ \vdots & & \vdots \\ \delta_\mu(\mathbf{x}^{\beta_1}) & \cdots & \delta_\mu(\mathbf{x}^{\beta_\mu}) \end{vmatrix}$$
for $\beta_i \in \mathbb{N}^{n+1}$, $|\beta_i| = d$ and $\beta_1 < \cdots < \beta_\mu$.

The $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ inside the $\mathcal{G}r^{\mu}_{S_d^*}(X)$

▶ Algebraic structure of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as projective variety is given by means of the bijection

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$W^A = \{(S_d^A/I_d, S_{d+1}^A/I_{d+1}) \in \mathcal{G}r^{\mu}_{S_d^{A}*}(X) \times \mathcal{G}r^{\mu}_{S_{d+1}^{A}*}(X) \mid S_1^A \cdot I_d = I_{d+1}\}.$$

The $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ inside the $\mathcal{G}r^{\mu}_{S_d^*}(X)$

- Algebraic structure of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as projective variety is given by means of the bijection

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$W^A = \{(S_d^A/I_d, S_{d+1}^A/I_{d+1}) \in \mathcal{G}r^{\mu}_{S_d^{A}*}(X) \times \mathcal{G}r^{\mu}_{S_{d+1}^{A}*}(X) \mid S_1^A \cdot I_d = I_{d+1}\}.$$

$$I_d \mapsto \overline{I_d} = (I_d) + (I_d : S_1) + (I_d : S_2) + \cdots + (I_d : S_{d-1})$$

The $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ inside the $\mathcal{G}r^{\mu}_{S_d^*}(X)$

- Algebraic structure of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as projective variety is given by means of the bijection

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$W^A = \{(S_d^A/I_d, S_{d+1}^A/I_{d+1}) \in \mathcal{G}r^{\mu}_{S_d^{A}{}^*}(X) \times \mathcal{G}r^{\mu}_{S_{d+1}^{A}{}^*}(X) \mid S_1^A \cdot I_d = I_{d+1}\}.$$

$$I_d \mapsto \overline{I_d} = (I_d) + (I_d : S_1) + (I_d : S_2) + \cdots + (I_d : S_{d-1})$$

- This holds by Gotzmann Persistence , and Regularity thms, and **There is an elementary proof by using border basis.**

The $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ inside the $\mathcal{G}r^{\mu}_{S_d^*}(X)$

- Algebraic structure of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ as projective variety is given by means of the bijection

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$W^A = \{(S_d^A/I_d, S_{d+1}^A/I_{d+1}) \in \mathcal{G}r^{\mu}_{S_d^{A}*}(X)\times\mathcal{G}r^{\mu}_{S_{d+1}^{A}*}(X) \mid S_1^A \cdot I_d = I_{d+1}\}.$$

$$I_d \mapsto \overline{I_d} = (I_d) + (I_d : S_1) + (I_d : S_2) + \cdots + (I_d : S_{d-1})$$

- This holds by Gotzmann Persistence , and Regularity thms, and **There is an elementary proof by using border basis.**
- In A-B-M (2008), Brachat-Lella-Mourrain-Roggero (2010), Lederer (?), find an inmersion of it inside the $\mathcal{G}r^{\mu}_{S_d^*}(X)$ with global equations of degree two. In the following we show how to get it inside a product of Grasmanians with equations of degree two.

Global equations for $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$

- I) **A Determinantal identity.**

Global equations for $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$

- I) **A Determinantal identity.** Let $\Delta := S_d^A / I_d \in \mathcal{G}r^{\mu}_{S_d^*}(X)$,

$B = (b_1, \ldots, b_\mu)$ be a family of homogeneous polynomials of degree d, then,

$$\Delta_B \, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}} \, b_i = 0 \text{ in } \Delta, \text{for } a \in S_d^A$$

where $B^{[b_i|a]} = (b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_\mu)$.

Global equations for $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$

- I) **A Determinantal identity.** Let $\Delta := S_d^A/I_d \in \mathcal{G}r^{\mu}_{S_d^*}(X)$,

$B = (b_1, \ldots, b_\mu)$ be a family of homogeneous polynomials of degree d, then,

$$\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0 \text{ in } \Delta, \text{for } a \in S_d^A$$

where $B^{[b_i|a]} = (b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_\mu)$. Let it be

$$M := \begin{bmatrix} \delta_1(a) & \delta_1(b_1) & \cdots & \delta_1(b_\mu) \\ \vdots & & & \vdots \\ \delta_\mu(a) & \delta_\mu(b_1) & \cdots & \delta_\mu(b_\mu) \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

As $M\mathrm{Adj}(M)^t = \det(M)\mathrm{I}_{(\mu+1)\times(\mu+1)}$. We get the last equality

$$M \begin{bmatrix} \Delta_B \\ \Delta_{B^{[b_1|a]}} \\ \vdots \\ \Delta_{B^{[b_\mu|a]}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \det(M) \end{bmatrix}.$$

Developing this product, the first μ coordinates show that every δ_j vanishes at $\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0$, therefore $\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0 \in \Delta$.

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$.

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$.

Proof. Let $(\Delta, \Delta^{'}) \in \mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ satisfying the equations above.

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta'_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta'_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B', x_k a$ is the family $(b'_1, \dots, b'_{\mu-1}, x_k a)$).

Proof. Let $(\Delta, \Delta') \in \mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ satisfying the equations above.

$\Delta = S_d / I_d$ with $ker(\Delta) := I$ satur. homog. ideal ofS_d.)

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$.

Proof. Let $(\Delta, \Delta^{'}) \in \mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ satisfying the equations above.
$\Delta = S_d/I_d$ with $ker(\Delta) := I$ satur. homog. ideal ofS_d.)
Let us to prove that $S_1 \cdot \ker \Delta \subset \ker \Delta^{'}$. Let B be a basis of Δ (so that Δ_B is invertible in A), and let f be an element of $\ker \Delta$.

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$.

Proof. Let $(\Delta, \Delta^{'}) \in \mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ satisfying the equations above. $\Delta = S_d/I_d$ with $ker(\Delta) := I$ satur. homog. ideal ofS_d.)

Let us to prove that $S_1 \cdot \ker \Delta \subset \ker \Delta^{'}$. Let B be a basis of Δ (so that Δ_B is invertible in A), and let f be an element of $\ker \Delta$.By linearity, equations above imply that $\Delta^{'}_{B',x_k f} = 0$ for all $k = 1, \ldots, n$ and all subset B' of $\mu - 1$ monomials of degree $d+1$ (because $\Delta_{B[b|f]} = 0$).

Conti.

Theorem: Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathcal{G}r^{\mu}_{S^*_d}(X)$ of the variety of $\mathcal{G}r^{\mu}_{S^*_d}(X) \times \mathcal{G}r^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

$$\Delta_B \, \Delta^{'}_{B',x_k a} - \sum_{b \in B} \Delta_{B[b|a]} \, \Delta^{'}_{B',x_k b} = 0,$$

for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B^{'}, x_k a$ is the family $(b^{'}_1, \ldots, b^{'}_{\mu-1}, x_k a)$.

Proof. The reciprocal argument is similar using the same determinantal equality.

Equations for $\mu = n = 2$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{(S^A_d/I_d, S^A_{d+1}/I_{d+1}) \in \mathcal{G}r^{\mu}_{S^A_d *}(X) \times \mathcal{G}r^{\mu}_{S^A_{d+1} *}(X) \mid S^A_1 \cdot I_d = I_{d+1}\}.$$

Its projection on $\mathcal{G}r^{\mu}_{S^A_d *}(X)$ gives the embedd. of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ in a Grasmanian.

Equations for $\mu = n = 2$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{(S^A_d/I_d, S^A_{d+1}/I_{d+1}) \in \mathcal{G}r^{\mu}_{S^A_d{}^*}(X) \times \mathcal{G}r^{\mu}_{S^A_{d+1}{}^*}(X) \mid S^A_1 \cdot I_d = I_{d+1}\}.$$

Its projection on $\mathcal{G}r^{\mu}_{S^A_d{}^*}(X)$ gives the embedd. of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ in a Grasmanian.The equations are computed with the same technique, but they are more involved.

Equations for $\mu = n = 2$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{(S^A_d/I_d, S^A_{d+1}/I_{d+1}) \in \mathcal{G}r^{\mu}_{S^{A}_{d}*}(X) \times \mathcal{G}r^{\mu}_{S^{A}_{d+1}*}(X) \mid S^A_1 \cdot I_d = I_{d+1}\}.$$

Its projection on $\mathcal{G}r^{\mu}_{S^{A}_{d}*}(X)$ gives the embedd. of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ in a Grasmanian.The equations are computed with the same technique, but they are more involved. **For $\mu = n = 2$ we obtain**, plus permutation of x, y and z:

Equations for $\mu = n = 2$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{(S^A_d/I_d, S^A_{d+1}/I_{d+1}) \in \mathcal{G}r^{\mu}_{S^{A}_{d}{}^{*}}(X) \times \mathcal{G}r^{\mu}_{S^{A}_{d+1}{}^{*}}(X) \mid S^A_1 \cdot I_d = I_{d+1}\}.$$

Its projection on $\mathcal{G}r^{\mu}_{S^{A}_{d}{}^{*}}(X)$ gives the embedd. of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ in a Grasmanian.The equations are computed with the same technique, but they are more involved. **For $\mu = n = 2$ we obtain**, plus permutation of x, y and z:

$$\left\{\begin{array}{l}
\Delta_{y^2,xy}\,\Delta_{x^2,xz} - \Delta_{xy,yz}\,\Delta_{x^2,xy} = 0,\\
\Delta_{y^2,xy}\,\Delta_{x^2,yz} - \Delta_{xy,xz}\,\Delta_{y^2,xy} - \Delta_{xy,yz}\,\Delta_{x^2,y^2} = 0,\\
\Delta_{x^2,xy}\,\Delta_{xy,xz} + \Delta_{x^2,y^2}\,\Delta_{x^2,xz} - \Delta_{x^2,yz}\,\Delta_{x^2,xy} = 0,\\
\Delta_{x^2,xy}\,\Delta_{z^2,xy} - \Delta^2_{xy,xz} - \Delta_{x^2,xz}\,\Delta_{xy,yz} = 0,\\
\Delta_{x^2,xy}\,\Delta_{x^2,z^2} - \Delta_{xy,xz}\,\Delta_{x^2,xz} - \Delta_{x^2,xz}\,\Delta_{x^2,yz} = 0,\\
\Delta_{z^2,xz}\,\Delta_{x^2,xy} - \Delta_{xz,yz}\,\Delta_{x^2,xz} = 0,\\
\Delta_{z^2,xz}\,\Delta_{x^2,zy} - \Delta_{xy,xz}\,\Delta_{z^2,xz} - \Delta_{zy,xz}\,\Delta_{x^2,z^2} = 0,\\
\Delta_{x^2,xz}\,\Delta_{xy,xz} + \Delta_{x^2,z^2}\,\Delta_{x^2,xy} - \Delta_{x^2,zy}\,\Delta_{x^2,xz} = 0,\\
\Delta_{x^2,xz}\,\Delta_{y^2,xz} - \Delta^2_{xy,xz} - \Delta_{x^2,xy}\,\Delta_{xz,yz} = 0,\\
\Delta_{x^2,xz}\,\Delta_{x^2,y^2} - \Delta_{xy,xz}\,\Delta_{x^2,xy} - \Delta_{x^2,xy}\,\Delta_{x^2,zy} = 0,\\
\Delta_{y^2,xz}\,\Delta_{yz,xz} + \Delta_{yz,xz}\,\Delta_{xy,yz} - \Delta_{yz,xz}\,\Delta_{y^2,xz} - \Delta_{z^2,xy}\,\Delta_{xy,y^2} = 0,\\
\Delta_{xy,y^2}\,\Delta_{yz,xz} + \Delta_{xy,yz}\,\Delta_{xy,yz} - \Delta_{xy,yz}\,\Delta_{y^2,xz} - \Delta_{xy,z^2}\,\Delta_{xy,y^2} = 0,\\
\Delta_{y^2,xz}\,\Delta_{z^2,xy} + \Delta_{yz,xz}\,\Delta_{xy,z^2} - \Delta_{yz,xz}\,\Delta_{yz,xz} - \Delta_{z^2,xy}\,\Delta_{xy,yz} = 0,\\
\Delta_{xy,y^2}\,\Delta_{z^2,xy} + \Delta_{xy,yz}\,\Delta_{xy,z^2} - \Delta_{xy,yz}\,\Delta_{yz,xz} - \Delta_{xy,z^2}\,\Delta_{xy,yz} = 0,\\
\Delta^2_{xy,xz} - \Delta_{xz,y^2}\,\Delta_{x^2,xz} - \Delta_{xz,yz}\,\Delta_{x^2,xy} = 0,\\
\Delta_{xz,yz}\,\Delta_{xz,x^2} + \Delta_{xz,z^2}\,\Delta_{xy,x^2} = 0,\\
\Delta^2_{xy,xz} - \Delta_{xy,yz}\,\Delta_{y^2,xz} - \Delta_{xy,z^2}\,\Delta_{xy,x^2} = 0.
\end{array}\right.$$

Equations for $\mu = n = 2$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) \longleftrightarrow$$

$$\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X) = \{(S^A_d/I_d, S^A_{d+1}/I_{d+1}) \in \mathcal{G}r^{\mu}_{S^A_d\,*}(X) \times \mathcal{G}r^{\mu}_{S^A_{d+1}\,*}(X) \mid S^A_1 \cdot I_d = I_{d+1}\}.$$

Its projection on $\mathcal{G}r^{\mu}_{S^A_d\,*}(X)$ gives the embedd. of $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ in a Grasmanian.The equations are computed with the same technique, but they are more involved. **For $\mu = n = 2$ we obtain**, plus permutation of x, y and z:

- They come from: commutation of the multiplication by the variables and by considering multiplication and changes of chart w.r.t the Plucker coordinates.

The same were obtained by Brodsky-Sturmfels (2010), using Groebner bases.

The Tangent space to the punctual Hilbert scheme

The Tangent space to the punctual Hilbert scheme

We consider the border relations at a point: $h^0_\alpha \ \alpha \in B$

The Tangent space to the punctual Hilbert scheme

We consider the border relations at a point: $h^0_\alpha \ \alpha \in B$

- Write $h^\varepsilon_\alpha = h^0_\alpha + \varepsilon h^1_\alpha + \mathcal{O}(\varepsilon^2)$ where $h^1_\alpha(\boldsymbol{x}) := \sum_{\beta \in B} h^1_{\alpha,\beta} \boldsymbol{x}^\beta$.

The Tangent space to the punctual Hilbert scheme

We consider the border relations at a point: h_α^0 $\alpha \in B$

- Write $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon h_\alpha^1 + \mathcal{O}(\varepsilon^2)$ where $h_\alpha^1(\boldsymbol{x}) := \sum_{\beta \in B} h_{\alpha,\beta}^1 \boldsymbol{x}^\beta$.

Determine the linear system satisfied by $\boldsymbol{h}^1 := (h_{\alpha,\beta}^1)_{\alpha \in \partial B, \beta \in B}$.

The Tangent space to the punctual Hilbert scheme

We consider the border relations at a point: $h_\alpha^0 \ \alpha \in B$

- Write $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon h_\alpha^1 + \mathcal{O}(\varepsilon^2)$ where $h_\alpha^1(\boldsymbol{x}) := \sum_{\beta \in B} h_{\alpha,\beta}^1 \boldsymbol{x}^\beta$.

Determine the linear system satisfied by $\boldsymbol{h}^1 := (h_{\alpha,\beta}^1)_{\alpha \in \partial B, \beta \in B}$.

- Operator *multiplication by* x_i : $M_{x_i}^\varepsilon$ decomposes:

$M_{x_i}^\varepsilon = M_{x_i}^0 + \varepsilon M_{x_i}^1 + \mathcal{O}(\varepsilon^2)$, where $M_{x_i}^0$ is the operator of multiplication by x_i in $\mathcal{A}^0$ and $M_{x_i}^1$ is linear in $\boldsymbol{h}^1$.

The Tangent space to the punctual Hilbert scheme

We consider the border relations at a point: h^0_α $\alpha \in B$

- Write $h^\varepsilon_\alpha = h^0_\alpha + \varepsilon h^1_\alpha + \mathcal{O}(\varepsilon^2)$ where $h^1_\alpha(\boldsymbol{x}) := \sum_{\beta \in B} h^1_{\alpha,\beta} \boldsymbol{x}^\beta$.

Determine the linear system satisfied by $\boldsymbol{h}^1 := (h^1_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$.

- Operator *multiplication by* x_i : $M^\varepsilon_{x_i}$ decomposes:

$M^\varepsilon_{x_i} = M^0_{x_i} + \varepsilon M^1_{x_i} + \mathcal{O}(\varepsilon^2)$, where $M^0_{x_i}$ is the operator of multiplication by x_i in $\mathcal{A}^0$ and $M^1_{x_i}$ is linear in $\boldsymbol{h}^1$.

The commutation implies

$$M^\varepsilon_{x_i} \circ M^\varepsilon_{x_j} - M^\varepsilon_{x_j} \circ M^\varepsilon_{x_i} = (M^0_{x_i} \circ M^0_{x_j} - M^0_{x_j} \circ M^0_{x_i}) +$$

$$+\varepsilon(M^1_{x_i} \circ M^0_{x_j} + M^0_{x_i} \circ M^1_{x_j} - M^1_{x_j} \circ M^0_{x_i} - M^0_{x_j} \circ M^1_{x_i}) + \mathcal{O}(\varepsilon^2)$$

$$= \varepsilon(M^1_{x_i} \circ M^0_{x_j} + M^0_{x_i} \circ M^1_{x_j} - M^1_{x_j} \circ M^0_{x_i} - M^0_{x_j} \circ M^1_{x_i}) + \mathcal{O}(\varepsilon^2)$$

- We deduce the linear equations in $\boldsymbol{h}^1$

$$M^1_{x_i} \circ M^0_{x_j} + M^0_{x_i} \circ M^1_{x_j} - M^1_{x_j} \circ M^0_{x_i} - M^0_{x_j} \circ M^1_{x_i} = 0 (1 \leqslant i < j \leqslant n)[***]$$

The above are the equations of the Tangent space T_{I_0} to the variety $\mathcal{H}_{\mathcal{B}}$ at the point I_0 whose border relations are $(h^0_\alpha)_\alpha$

THANK YOU FOR YOUR ATTENTION!