Combinatorics of ideals of points: a Cerlienco-Mureddu-like approach for an iterative lex game.

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ACA2018
Santiago de Compostela
19th June 2018

Once upon a time...

Cerlienco-Mureddu (1990)

Given a finite set of distinct points $\mathbf{X}$, compute the lexicographical Groebner escalier $\mathrm{N}(\mathrm{I}(\mathbf{X}))$ of the ideal of the points $I(\mathbf{X})$.

There is a $1-1$ correspondence between $\mathbf{X}$ and $\mathrm{N}(\mathrm{I}(\mathbf{X}))$.

The algorithm providing the correspondence is **iterative** on the points and **inductive** on the variables.
Complexity: n^2S^2 ($n =$number of variables, $S = |\mathbf{X}|$).

An improvement: the lex game

Felszeghy-Rath-Ronyai (2006)

By a clever use of tries (point trie - lex trie), they develop an algorithm that computes the lexicographical escalier in a more efficient way.

The algorithm **drops iterativity** for the sake of efficiency.
Complexity: $nS + S\min(S, nr)$
($r =$ maximal number of children of a node).

The point trie

It is a trie representing the reciprocal relations among the coordinates of points.

same path from level 0 to level $i =$ same $1, ..., i$ first coordinates

It is constructed **iteratively** on the points.

Example of point trie

$$\mathbf{X} = \{P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (1,1,2), P_4 = (1,0,3)\}$$

$$\{1\}$$
$$1 \;|$$
$$\{1\}$$
$$0 \;|$$
$$\{1\}$$
$$0 \;|$$
$$\{1\}$$

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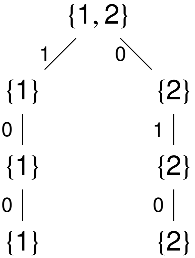

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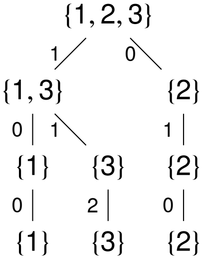

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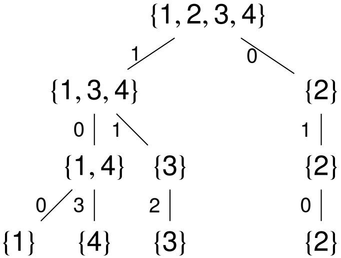

Another point of view: Moeller's algorithm

Moeller (1993)

Given an ordered finite set of distinct points $\mathbf{X} := \{P_1, ..., P_S\}$, find, *for each ideal in Macaulay's chain* $I_i := I(\{P_1, ..., P_i\})\ 1 \leq i \leq S$, the **escalier** $\mathrm{N}(\mathrm{I}_i)$ and a **separator family** for the points (with some more steps you also get the Groebner bases).

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→ iterative on points

→ the result (for Lex) is exactly that of Cerlienco-Mureddu algorithm.

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→ iterative on points
→ the result (for Lex) is exactly that of Cerlienco-Mureddu algorithm.

Mora

With the same input data, if we make an adaptation of Moeller algorithm (some more computations, and keeping track of some more information) we can get more information, such as Groebner representation and Auzinger-Stetter matrices and the complexity is actually the same

Can we construct a new algorithm, that is

iterative as Cerlienco-Mureddu

and has the

same complexity as the lex game?

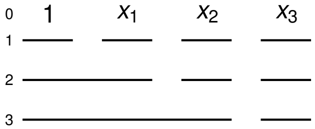
Bar Codes

Definition

A Bar Code B is a picture composed by segments, called *bars*, superimposed in horizontal rows, which satisfies

A. $\forall i, j,\ 1 \leq i \leq n-1,\ 1 \leq j \leq \mu(i),\ \exists! \bar{j} \in \{1, ..., \mu(i+1)\}$ s.t. $B_{\bar{j}}^{(i+1)}$ lies under $B_j^{(i)}$

B. $\forall i_1,\ i_2 \in \{1, ..., n\},\ \sum_{j_1=1}^{\mu(i_1)} l_1(B_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(B_{j_2}^{(i_2)})$; we will then say that *all the rows have the same length*.

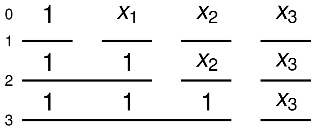


Associating monomials to bars

For $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T}$, $\forall i \in \{1, ..., n\}$, $\pi^i(t) := x_i^{\gamma_i} \cdots x_n^{\gamma_n}$;
$M = \{t_1, ..., t_m\} \subset \mathcal{T}$, $M^{[i]} := \pi^i(M)$, $\underline{M}$, $\underline{M}^{[i]}$ increasingly ordered w.r.t. Lex.

$$\mathcal{M} := \begin{pmatrix} \pi^1(t_1) & ... & \pi^1(t_m) \\ \pi^2(t_1) & ... & \pi^2(t_m) \\ \vdots & & \vdots \\ \pi^n(t_1) & ... & \pi^n(t_m) \end{pmatrix}$$

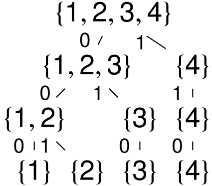
Bar Code: connecting with a bar the repeated terms

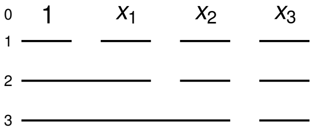


Bar Code and point trie

We can see the Bar Code as a **point trie** by taking as points the **exponents' lists** (→ **Macaulay's trick**) for the given terms.

For $M = \{1, x_1, x_2, x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we have $\mathfrak{M} = \{p_1 = (0,0,0), p_2 = (0,0,1), p_3 = (0,1,0), p_4 = (1,0,0)\}$, so we have





Several applications of Bar Code

Bar Codes are useful to study properties of monomial/polynomial ideals:

- counting **(strongly) stable ideals**;
- computing **Pommaret bases** via interpolation;
- computing **Janet multiplicative variables** and **Janet-like multiplicative powers**
- detect **completeness**;
- find variables' orderings which make a set of terms **Janet-complete**
- Bar Code, point trie vs. **Janet tree**

Two questions

In St Petersburg...

1. *Lundqvist:* since the BC is similar to the lex/point trie; why do you use it instead of using the tries?

   Mora: since we cannot give an iterative version of the Lex Game using tries

   Lundqvist: ok, I will try to find an iterative algorithm with the same complexity using tries.

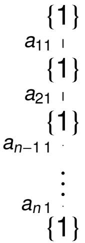
2. *Robbiano* can you generalize your algorithm to the case of multiple points?

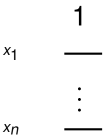
   Mora: probably not, since Macaulay's language is heavy...

   Ceria-Mora: we have to do it!! $\rightarrow$ research in progress by generalizing Lundqvist's results (see previous talk)

Base step

$|\mathbf{X}| = N = 1$: set $\mathrm{N}(1) = \{1\}$ and construct the point trie $T(P_1) = \mathfrak{T}(\mathbf{X})$ and the Bar Code $\mathrm{B}(1)$ displayed below. The output is stored in the matrix M.





$$M = \begin{bmatrix} & \mathbf{x_n} & \mathbf{x_{n-1}} & \dots & \mathbf{x_1} \\ & \downarrow & \downarrow & \dots & \downarrow \\ \mathbf{1} \rightarrow & 0 & 0 & \dots & 0 \end{bmatrix}$$

Our algorithm: $|\mathbf{X}| = N > 1$

- update the point trie: forking level $s = \sigma$**-value**; leftmost label of the rightmost sibling $l = \sigma$**-antecedent**;
- find the s–bar of t_l: $\mathrm{B}_j^{(s)}$

Information on t_N:

- it lies over $\mathrm{B}_1^{(n)}, \mathrm{B}_1^{(n-1)}, ..., \mathrm{B}_1^{(s+1)}$ so t_N lies over the first $n, ..., s+1$ bars, i.e. $a_{s+1}^{(N)} = ... = a_n^{(N)} = 0$, so $x_n, ..., x_{s+1} \nmid t_N$;
- it should lie over $\mathrm{B}_{j+1}^{(s)}$: $a_s^{(N)} = a_s^{(l)} + 1$.

OUR ALGORITHM: $|\mathbf{X}| = N > 1$

We test whether $B_{j+1}^{(s)}$ lies over $B_1^{(n)}, B_1^{(n-1)}, ..., B_1^{(s+1)}$; two possible cases

A. **NO**: we construct a new s-bar of lenght one over $B_1^{(n)}, B_1^{(n-1)}, ..., B_1^{(s+1)}$, on the right of $B_j^{(s)}$, we label it as $B_{j+1}^{(s)}$ and we construct a $1, ..., s-1$ bar of length 1 over $B_{j+1}^{(s)}$: $t_N = x_s^{j+2}$; store the output in the N-th row of M.

B. **YES**: we must continue, repeating the procedure

Our algorithm: $|\mathbf{X}| = N > 1$

- *restrict* the point trie *to the points* whose corresponding terms lie over $\mathsf{B}^{(s)}_{j+1}$. The set containing these points is denoted by S and is *obtained reading* $\mathsf{B}^{(s)}_{j+1}$. More precisely, $S = \psi(\mathsf{B}^{(s)}_{j+1})$, where

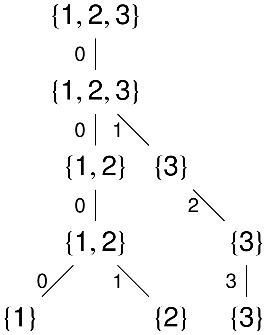
$$\psi : \mathsf{B} \to \mathcal{T}$$

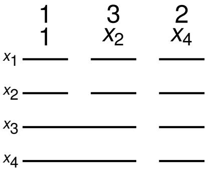
  is the function sending each 1-bar $\mathsf{B}^{(1)}_{l}$ in the term t_l over it and, inductively, for $1 < u \leq n$, $\psi(\mathsf{B}^{(u)}_{h}) = \bigcup_{B \text{ over } \mathsf{B}^{(u)}_{h}} \psi(B)$
- *read P_N's path*, from level $s - 1$ to level 1, *looking for the first forking level w.r.t. S* (σ-value/σ-antecedent as before).
- **repeat** the test

The procedure is repeated until we get to the 1-bars or if in the decision step we get case a.

Example

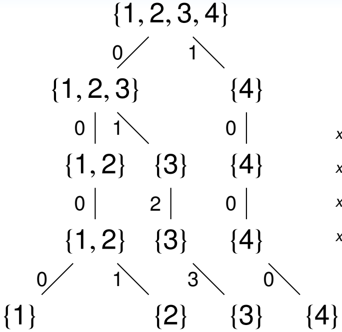
$\mathbf{X} = \{P_1 = (0,0,0,0), P_2 = (0,0,0,1), P_3 = (0,1,2,3), P_4 = (1,0,0,0), P_5 = (1,0,0,1)\}$

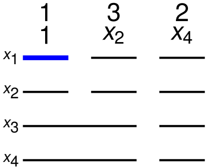




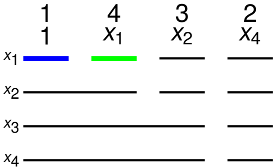
$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For $P_4 = (1,0,0,0)$, $s = 1$, $l = 1$; B the blue bar

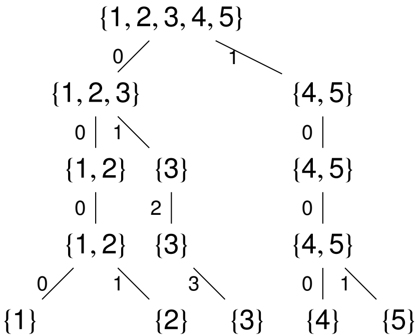




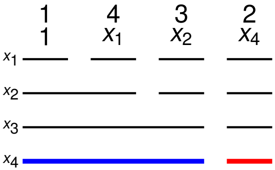
There is no 1-bar on the right of B, lying over $\mathrm{B}_1^{(4)}, \mathrm{B}_1^{(3)}, \mathrm{B}_1^{(2)}$:



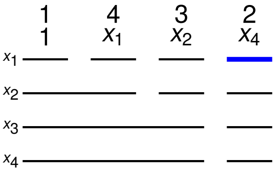
$P_5 = (1, 0, 0, 1)$; $s = 4$ $l = 4$:



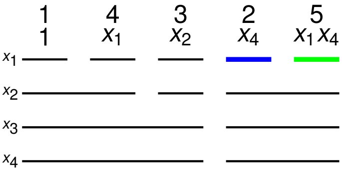
$B = \mathrm{B}_1^{(4)}$; $B' = \mathrm{B}_2^{(4)}$, $S = \{P_2\}$.



The fork with P_2 happens at $s = 1$ and the σ-antecedent is P_l, for $l = 2$, so $B = \mathrm{B}_4^{(1)}$.



Since B' still does not exist, we create it



$\mathrm{N} = \{1, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_4, \mathrm{x}_1\mathrm{x}_4\}$

Separator polynomials

A **family of separators** for a finite set $\mathbf{X} = \{P_1, ..., P_N\}$ of distinct points is a set $Q = \{Q_1, ..., Q_N\}$ s.t.
$Q_i(P_i) = 1$ and $Q_i(P_j) = 0$, for each $1 \leq i, j \leq N$, $i \neq j$.

$\mathbf{X} = \{P_1, ..., P_N\}$, with $P_i := (a_{1,i}, ..., a_{n,i})$, $i = 1, ..., N$, we denote by $C = (c_{i,j})$ the **witness matrix** i.e. the (symmetric) matrix s.t., for $i, j = 1, ..., N$, $c_{i,j} = 0$ if $i = j$ and if $i \neq j$,
$c_{i,j} = \min\{h : 1 \leq h \leq n \text{ s.t. } a_{h,i} \neq a_{h,j}\}$.

Building blocks:

$$p_{i,j}^{[c_{i,j}]} = \frac{x_{c_{i,j}} - a_{c_{i,j},j}}{a_{c_{i,j},i} - a_{c_{i,j},j}}$$

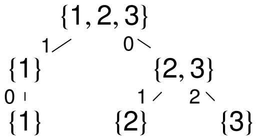
$|\mathbf{X}| = 1$: $Q_1 = 1$. $Q_1, ..., Q_{N-1}$ associated to $\{P_1, ..., P_{N-1}\}$: P_N?
We see now how to get the new separators $Q'_1, ..., Q'_N$ for $\mathbf{X}$.

- Set $Q'_N = 1$.
- $\forall j = 1, ..., n$, we take the node $v_{j,u}$ of N
- for each sibling $v_{j,u'}$ of $v_{j,u}$, we pick an element $\bar{i}$ of its label and set $Q'_N = Q'_N p^{[j]}_{N,\bar{i}}$.
- if $v_{j,u}$ is labelled only by N, then, for each sibling $v_{j,u'}$, for each element i in its label we set $Q'_i = Q_i p^{[j]}_{i,N}$.

Once concluded this procedure, if a separator Q_h, $1 \leq h \leq N$ has **not** been involved in the above steps, we set $Q'_h = Q_h$, getting a family of separators $\{Q'_1, ..., Q'_N\}$ for $\mathbf{X} = \{P_1, ..., P_N\}$.
Complexity of a single iterative round: $\mathcal{O}(\min(N, nr))$.

Example

$\mathbf{X} = \{P_1 = (1,0), P_2 = (0,1), P_3 = (0,2)\}$



In the first step, we set $Q_1'' = 1$; then, adding P_2 to the trie we set $Q_2' = p_{2,1}^{[1]} = -(x-1)$ and we modify also Q_1'', setting $Q_1' = Q_1'' p_{1,2}^{[1]} = x$, since, when P_3 is still not in the trie, the node $v_{1,2}$, has $V_{1,2} = \{2\}$. So, w.r.t. $\{P_1, P_2\}$, we have $Q_1' = x$, $Q_2' = -(x-1)$. Finally, we add P_3. This way, $Q_3 = p_{3,1}^{[1]} p_{3,2}^{[2]} = -(x-1)(y-1)$ and since $V_{2,3} = \{3\}$, $Q_2 = Q_2' p_{2,3}^{[2]} = (x-1)(y-2)$. Finally, we have

$$Q_1 = x; \; Q_2 = (x-1)(y-2); \; Q_3 = -(x-1)(y-1).$$

Comparisons?

$$Q_1 = x;\ Q_2 = (x-1)(y-2);\ Q_3 = -(x-1)(y-1).$$

From Lex game

$$Q_1 = \frac{1}{2}x(y-1)(y-2);\ Q_2 = y(x-1)(y-2);\ Q_3 = -\frac{1}{2}(x-1)y(y-1),$$

Lundqvist

$$Q_1 = x^2;\ Q_2 = (x-1)(y-2);\ Q_3 = -(x-1)(y-1).$$

Moeller

$$Q_1 = x;\ Q_2 = 2-2x-y;\ Q_3 = x+y-1.$$

Auzinger-Stetter

$I \triangleleft \mathbf{k}[x_1, ..., x_n]$ zerodimensional ideal; $A := \mathbf{k}[x_1, ..., x_n]/I$. $\forall f \in A$, $\Phi_f : A \to A$ multiplication by f in A and, fixed a basis $B = \{[b_1], \ldots, [b_m]\}$ for A, $A_f = (a_{ij})$ so that

$$[b_i f] = \sum_j a_{ij}[b_j], \forall i.$$

We call **Auzinger-Stetter matrices** associated to I, the matrices A_{x_i}, $i = 1, ..., n$, defined w.r.t. the basis given by the lex escalier of I.

Lundqvist

$\mathbf{X} = \{P_1, ..., P_N\}$, $I := I(\mathbf{X}) \triangleleft \mathbf{k}[x_1, ..., x_n]$; $\mathrm{N} = \{\mathrm{t}_1, ..., \mathrm{t}_\mathrm{N}\} \subset \mathbf{k}[\mathrm{x}_1, ..., \mathrm{x}_\mathrm{n}]$ s.t. $[\mathrm{N}] = \{[\mathrm{t}_1], ..., [\mathrm{t}_\mathrm{N}]\}$ is a basis for $A := \mathbf{k}[x_1, ..., x_n]/I$. Then, for each $f \in \mathbf{k}[x_1, ..., x_n]$ we have

$$\mathbf{Nf}(f, \mathrm{N}) = (\mathrm{t}_1, ..., \mathrm{t}_\mathrm{N})(\mathrm{N}(\mathbf{X})^{-1})^t(\mathrm{f}(\mathrm{P}_1), ..., \mathrm{f}(\mathrm{P}_\mathrm{N}))^t,$$

where $\mathrm{Nf}(f, \mathrm{N})$ is the normal form of f w.r.t. N.

Notation

- $A_{x_h} := \left(a_{li}^{(h)}\right)_{li}$, $1 \leq h \leq n$, $1 \leq l, i \leq N$, the Auzinger-Stetter matrices w.r.t. $\mathrm{N}(\mathbf{I})$;
- $B := \mathrm{N}(\mathbf{I})(\mathbf{X}) := (\mathrm{b}_{\mathrm{lj}})_{\mathrm{lj}}$, $1 \leq l, j \leq N$, $b_{lj} := t_l(P_j)$;
- $C := (c_{ji})_{ji}$, $1 \leq j, i \leq N$, the inverse matrix of B, i.e. $C := B^{-1}$
- $D^{(h)} := \left(d_{lj}^{(h)}\right)_{lj}$, $1 \leq h \leq n$, $1 \leq l, j \leq N$, $d_{lj}^{(h)} := \alpha_h^{(j)} t_l(P_j)$, the evaluation of $x_h t_l$ at the point P_j.

Lundqvist

$\mathbf{X} = \{P_1, ..., P_N\}$, $I := I(\mathbf{X}) \triangleleft \mathbf{k}[x_1, ..., x_n]$; $\mathrm{N} = \{\mathrm{t}_1, ..., \mathrm{t_N}\} \subset \mathbf{k}[\mathrm{x}_1, ..., \mathrm{x_n}]$ s.t. $[\mathrm{N}] = \{[\mathrm{t}_1], ..., [\mathrm{t_N}]\}$ is a basis for $A := \mathbf{k}[x_1, ..., x_n]/I$. Then, for each $f \in \mathbf{k}[x_1, ..., x_n]$ we have

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where $\mathrm{Nf}(f, \mathrm{N})$ is the normal form of f w.r.t. N.

For $1 \leq l \leq N$, the l-th row of A_{x_h} is the normal form of $x_h t_l$:

$$\mathrm{Nf}(x_h t_l, \mathrm{N(I)}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{a_{li} t_i} = (\mathrm{t}_1, ..., \mathrm{t_N})\mathrm{C}^{\mathrm{t}}(\mathrm{x_h t_l}(\mathrm{P}_1), ..., \mathrm{x_h t_l}(\mathrm{P_N}))^{\mathrm{t}} =$$

$$(t_1, ..., t_N)C^t(d_{l1}^{(h)}, ..., d_{lN}^{(h)})^t = \sum_i \left(\sum_{j=1}^{N} d_{lj}^{(h)} c_{ji} \right) t_i.$$

This trivially implies that $A_{x_h} = D^{(h)}C = D^{(h)}B^{-1}$.

Computing B^{-1}.

Gaussian column-reduction of $\begin{pmatrix} B \\ \hline I \end{pmatrix}$.

At each step

$$\begin{pmatrix} B \\ \hline I \end{pmatrix} \to \begin{pmatrix} E \\ \hline F \end{pmatrix}$$

it holds $E = BF$ So $E = 1 \implies F = B^{-1}$.

We border B obtaining $B' := \begin{pmatrix} & B & & b_{1N} \\ & & & \vdots \\ & & & b_{N-1N} \\ b_{N1} & \cdots & b_{NN-1} & b_{NN} \end{pmatrix}$ and

properly border $\left(\dfrac{I}{C}\right)$ as $\left(\begin{array}{cccc} & I & & b_{1N} \\ & & & \vdots \\ & & & b_{N-1N} \\ f_{N1} & \cdots & f_{NN-1} & b_{NN} \\ \hline & & & 0 \\ & C & & \vdots \\ & & & 0 \\ 0 & \cdots 0 & & 1 \end{array}\right)$ where

$$(f_{N1} \cdots f_{NN-1}) = (b_{N1} \cdots b_{NN-1})\, C$$

For each point i we know the last σ-value $s(i)$ and σ-antecedent $P_{l(i)}$ $\boxed{t_i = x_{s(i)} t_{l(i)}}$

We perform the following computations

- $b_{1N} := 1$
- for $i = 2 \cdots N - 1$, $b_{iN} := b_{l(i)N} a_{s(i)N}$
- for $j = 1 \cdots N$, $b_{Nj} := b_{l(N)j} a_{s(N)N}$
  border B

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- $b_{1N} := 1$
- for $i = 2 \cdots N-1, b_{iN} := b_{l(i)N} a_{s(i)N}$
- for $j = 1 \cdots N, b_{Nj} := b_{l(N)j} a_{s(N)N}$
  border B
- for $i = 1 \cdots N-1, 1 \leq h \leq n, d_{iN}^{(h)} := d_{l(i)N}^{(h)} a_{s(i)N}$
- for $j = 1 \cdots N, 1 \leq h \leq n, d_{Nj}^{(h)} := d_{l(N)j}^{(h)} a_{s(N)N}$
  border D

For each point i we know the last σ-value $s(i)$ and σ-antecedent $P_{l(i)}$ $\boxed{t_i = x_{s(i)} t_{l(i)}}$

We perform the following computations

- $b_{1N} := 1$
- for $i = 2 \cdots N-1, b_{iN} := b_{l(i)N} a_{s(i)N}$
- for $j = 1 \cdots N, b_{Nj} := b_{l(N)j} a_{s(N)N}$
  border B
- for $i = 1 \cdots N-1, 1 \leq h \leq n, d_{iN}^{(h)} := d_{l(i)N}^{(h)} a_{s(i)N}$
- for $j = 1 \cdots N, 1 \leq h \leq n, d_{Nj}^{(h)} := d_{l(N)j}^{(h)} a_{s(N)N}$
  border D
- for $i = 1 \cdots N-1, f_{Ni} := \sum_j b_{Nj} c_{ji}$
  border C

- for $i = 1 \cdots N - 1, g_{iN} := \sum_j c_{ij} b_{jN}$
- $h_{NN} := f_{NN} - \sum_j f_{Nj} b_{jN}$
- $c_{iN} := \frac{g_{iN}}{h_{NN}}, 1 \leq i \leq N$
- $c_{ij} := c'_{ij} - f_{Nj} c_{iN} 1 \leq i \leq N, 1 \leq j < N$

  computing $C = B^{-1}$

- for $i = 1 \cdots N-1$, $g_{iN} := \sum_j c_{ij} b_{jN}$
- $h_{NN} := f_{NN} - \sum_j f_{Nj} b_{jN}$
- $c_{iN} := \frac{g_{iN}}{h_{NN}}, 1 \leq i \leq N$
- $c_{ij} := c'_{ij} - f_{Nj} c_{iN} 1 \leq i \leq N, 1 \leq j < N$

  computing $C = B^{-1}$
- for $1 \leq l < N, 1 \leq h \leq n$, $a_{lN}^{(h)} := \sum_i d_{li}^{(h)} c_{iN}$,
- for $1 \leq j < N, 1 \leq h \leq n$, $a_{Nj}^{(h)} := \sum_i d_{Ni}^{(h)} c_{ij}$,

  $A^{(h)} = CD^{(h)}$

Example

For $\mathbf{X} = \{P_1 = (1,0), P_2 = (0,1), P_3 = (0,2)\}$:

P_1: $B = C = 1$ and $D^{(1)} = (1) = A_x$, $D^{(2)} = (0) = A_y$.

P_2: $B'' = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} I'' \\ \hline C'' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{pmatrix} \to B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, C = B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$$D^{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A_x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, D^{(2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_y = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

P_3: $B'' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, C'' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I'' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \to$

$$C = B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}. D^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D^{(2)} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 4 \end{pmatrix}, A_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & -2 & 3 \end{pmatrix}.$$

Thank you for your attention!