Computing Subschemes of the Border Basis Scheme

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Applications of Computer Algebra - ACA 2018

Santiago de Compostela, 19.6.2018

Contents

Contents

Contents

Contents

Contents

Contents

Contents

Contents

This is joint work with

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$R = P/I$ affine coordinate ring of $\mathbb{X}$

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(c) Let $\gamma_{ij} \in K$. Then the set $G = \{g_1, \ldots, g_\nu\}$ such that $g_j = b_j - \sum_{i=1}^{\mu} \gamma_{ij}\, t_i$ is called an **$\mathcal{O}$-border prebasis**.

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(d) An $\mathcal{O}$-border prebasis G is called an **$\mathcal{O}$-border basis** of I if $I = \langle G \rangle$ and if $\mathcal{O}$ represents a K-basis of $R = P/I$.

Picture of an order ideal and its border

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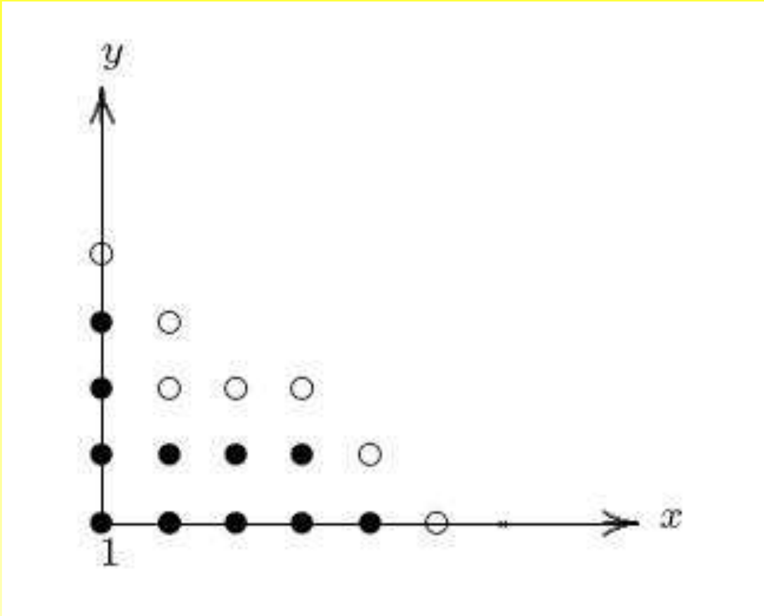

• term in the order ideal ◦ term in the border

Definition 1.2 Let G be an $\mathcal{O}$-border prebasis as above. For $r = 1, \ldots, n$, the matrix $\mathcal{A}_r = (a_{ij}^{(r)}) \in \mathrm{Mat}_\mu(K)$, where

$$a_{ij}^{(r)} = \begin{cases} \delta_{im} & \text{if } x_r t_j = t_m \\ \gamma_{im} & \text{if } x_r t_j = b_m \end{cases}$$

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Theorem 1.3 (Mourrain)
An $\mathcal{O}$-border prebasis $G \subset I$ is an $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices **commute**, *i.e. if and only if $\mathcal{A}_i \mathcal{A}_j - \mathcal{A}_j \mathcal{A}_i = 0$ for $1 \le i < j \le n$.*

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Definition 2.1 Let c_{ij} be indeterminates. Then the set $G = \{g_1, \ldots, g_\nu\}$ such that $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}\, t_i$ is called the **generic $\mathcal{O}$-border prebasis**.

Definition 2.2 **(a)** For $r = 1, \ldots, n$, the matrix $\mathcal{A}_r = (a_{ij}^{(r)}) \in \mathrm{Mat}_\mu(K[c_{ij}])$, where

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(b) Consider the ideal in $K[c_{ij}]$ which is generated by all entries of the commutator matrices $\mathcal{A}_r\mathcal{A}_s - \mathcal{A}_s\mathcal{A}_r$ with $1 \le r < s \le n$. Then the subscheme of $\mathbb{A}_K^{\mu\nu} = \mathrm{Spec}(K[c_{ij}])$ defined by this ideal is called the **$\mathcal{O}$-border basis scheme**. It is denoted by $\mathbb{B}_\mathcal{O}$, its vanishing ideal is denoted by $I(\mathbb{B}_\mathcal{O})$, and its affine coordinate ring is denoted by $B_\mathcal{O} = K[c_{11}, \ldots, c_{\mu\nu}]/I(\mathbb{B}_\mathcal{O})$.

Example 2.3 Let $\mathcal{O} = \{1, x, y, xy\} \subseteq \mathbb{T}^2$. Then we have

$$\mathcal{A}_x = \begin{pmatrix} 0 & c_{12} & 0 & c_{14} \\ 1 & c_{22} & 0 & c_{24} \\ 0 & c_{32} & 0 & c_{34} \\ 0 & c_{42} & 1 & c_{44} \end{pmatrix} \quad \text{and} \quad \mathcal{A}_y = \begin{pmatrix} 0 & 0 & c_{11} & c_{13} \\ 0 & 0 & c_{21} & c_{23} \\ 1 & 0 & c_{31} & c_{33} \\ 0 & 1 & c_{41} & c_{43} \end{pmatrix}$$

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and the defining ideal of $\mathbb{B}_{\mathcal{O}}$ is generated by

$$\begin{array}{ll} \{\ c_{11}c_{32} + c_{13}c_{42} - c_{14}, & c_{12}c_{21} + c_{14}c_{41} - c_{13}, \\ c_{21}c_{32} + c_{23}c_{42} - c_{24}, & c_{12}c_{23} - c_{11}c_{34} + c_{14}c_{43} - c_{13}c_{44}, \\ c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23}, & c_{23}c_{32} - c_{31}c_{34} + c_{34}c_{43} - c_{33}c_{44} - c_{14}, \\ c_{31}c_{32} + c_{33}c_{42} + c_{12} - c_{34}, & c_{22}c_{23} - c_{21}c_{34} + c_{24}c_{43} - c_{23}c_{44} + c_{13}, \\ c_{21}c_{32} + c_{34}c_{41} - c_{33}, & c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}, \\ c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}, & c_{34}c_{41} - c_{23}c_{42} + c_{24} - c_{33}\ \} \end{array}$$

Remark 2.4 **(a)** The border basis scheme is an open subscheme of the Hilbert scheme $\mathrm{Hilb}^{\mu}(\mathbb{A}^n)$ parametrizing all 0-dimensional subschemes of $\mathbb{A}^n$ of length μ.

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(c) The various border basis schemes for order ideals with μ elements cover the Hilbert scheme.

Idea: Using the generic multiplication matrices and the algorithms given in the preceding talk, we can calculate sets of equations which define **subschemes of** $\mathbb{B}_{\mathcal{O}}$ parametrizing 0-dimensional schemes having certain special properties such as Gorenstein schemes, CBP, strict Gorenstein schemes, strict complete intersections, etc.

3 – Computing the Locally Gorenstein Locus

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Definition 3.1 The set of all K-rational points $\Gamma = (\gamma_{ij}) \in K^{\mu\nu}$ of the border basis scheme $\mathbb{B}_{\mathcal{O}}$ whose associated 0-dimensional scheme $\mathbb{X}_\Gamma$ is locally Gorenstein is called the **locally Gorenstein locus** of $\mathbb{B}_{\mathcal{O}}$ and is denoted by $\mathrm{LGor}(\mathcal{O})$.

Algorithm 3.2 **(The Non-Locally Gorenstein Locus in $\mathbb{B}_{\mathcal{O}}$)**

The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathbb{B}_{\mathcal{O}}$.

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(2) Calculate the commutators $\mathcal{A}_r\mathcal{A}_s - \mathcal{A}_s\mathcal{A}_r$ for $1 \leq r < s \leq n$ and form the ideal $I(\mathbb{B}_{\mathcal{O}})$ in $K[c_{ij}]$ generated by their entries.

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(3) Introduce new indeterminates $z_1, \ldots, z_\mu$ and construct the matrix C in $\mathrm{Mat}_\mu(K[c_{ij}][z_1, \ldots, z_\mu])$ whose i-th column is given by $t_i(\mathcal{A}_1^{\,\mathrm{tr}}, \ldots, \mathcal{A}_n^{\,\mathrm{tr}}) \cdot (z_1, \ldots, z_\mu)^{\,\mathrm{tr}}$.

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(4) Compute $\det(C)$ in $K[c_{ij}][z_1, \ldots, z_\mu]$, and let J be the ideal in $K[c_{ij}]$ generated by the coefficients of $\det(C)$ w.r.t. $z_1, \ldots, z_\mu$.

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(5) Return the ideal $I(\mathbb{B}_{\mathcal{O}}) + J$.

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Let $\mathcal{Z} = (z_1, z_2, z_3, z_4)^{\mathrm{tr}}$ and form the matrix $C = (\mathcal{Z},\ \mathcal{A}_x\mathcal{Z},\ \mathcal{A}_y\mathcal{Z},\ \mathcal{A}_x\mathcal{A}_y\mathcal{Z})$. Its four columns are

$$\mathcal{Z},\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_{12} & c_{22} & c_{32} & c_{42} \\ 0 & 0 & 0 & 1 \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix}\mathcal{Z},\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{11} & c_{21} & c_{31} & c_{41} \\ c_{13} & c_{23} & c_{33} & c_{43} \end{pmatrix}\mathcal{Z},\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ p_1 & p_2 & p_3 & p_4 \\ c_{13} & c_{23} & c_{33} & c_{43} \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}\mathcal{Z}$$

where $p_1 = c_{11}c_{32} + c_{13}c_{42}$, $p_2 = c_{21}c_{32} + c_{23}c_{42}$, $p_3 = c_{31}c_{32} + c_{33}c_{42} + c_{12}$, $p_4 = c_{32}c_{41} + c_{42}c_{43} + c_{22}$, $q_1 = c_{11}c_{34} + c_{13}c_{44}$, $q_2 = c_{21}c_{34} + c_{23}c_{44}$, $q_3 = c_{31}c_{34} + c_{33}c_{44} + c_{14}$, and $q_4 = c_{34}c_{41} + c_{43}c_{44} + c_{24}$.

The determinant of C is a polynomial

$$\det(C) = (-c_{11}^2 c_{14} c_{32} + c_{11}^2 c_{12} c_{34} - c_{11} c_{13} c_{14} c_{42} + c_{11} c_{12} c_{13} c_{44} - c_{12} c_{13}^2)\, z_1^4 + \cdots + (-c_{41} c_{42} + 1)\, z_4^4$$

in $K[c_{ij}][z_1, z_2, z_3, z_4]$ which is homogeneous of degree 4 with respect to $z_1, \ldots, z_4$ and has 35 non-zero coefficients in $K[c_{ij}]$. Let J be the ideal generated by these coefficients. Then the **Non-Locally Gorenstein Locus** $\mathrm{NonLGor}(\mathcal{O})$ is defined by the ideal $I(\mathbb{B}_{\mathcal{O}}) + J$.

Via the isomorphism $B_{\mathcal{O}} \cong \widetilde{P} = K[c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]$, we can examine $\mathrm{NonLGor}(\mathcal{O})$ further. Let $\widetilde{J}$ be the image of J in $\widetilde{P}$. Then we can compute a Gröbner basis of $\widetilde{J}$ and check that $\dim(\widetilde{P}/\widetilde{J}) = 4$. Hence $\mathrm{NonLGor}(\mathcal{O})$ is the set of closed points of a 4-dimensional closed subscheme of $\mathbb{B}_{\mathcal{O}} \cong \mathbb{A}^8$.

4 – The Degree Filtered Border Basis Scheme

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Recall that the **degree filtration** of $R = P/I$ is given by $F_i R = P_{\leq i}/(I \cap P_{\leq i})$ for $i \in \mathbb{Z}$.

Definition 4.1 **(a)** A tuple $B = (\bar{t}_1, \ldots, \bar{t}_\mu) \in R^\mu$ is called a **degree filtered K-basis** of R if the set $B \cap F_i R$ is a K-basis of $F_i R$ for every $i \in \mathbb{Z}$ and if $\mathrm{ord}(\bar{t}_1) \leq \cdots \leq \mathrm{ord}(\bar{t}_\mu)$.

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(b) We say that I has a **degree filtered $\mathcal{O}$-border basis** if $\overline{\mathcal{O}}$ is a degree filtered K-basis of R.

Proposition 4.2 *For a K-rational point $\Gamma = (\gamma_{ij})$ of $\mathbb{B}_{\mathcal{O}}$, the 0-dimensional scheme $\mathbb{X}_\Gamma$ associated to Γ has a degree filtered $\mathcal{O}$-border basis if and only if $\gamma_{ij} = 0$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ such that $\deg(t_i) > \deg(b_j)$.*

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Definition 4.3 Let $I_{\mathcal{O}}^{\mathrm{df}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates c_{ij} such that $\deg(t_i) > \deg(b_j)$.

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(a) The closed subscheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$ of $\mathbb{B}_{\mathcal{O}}$ defined by $I(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}) = I(\mathbb{B}_{\mathcal{O}}) + I_{\mathcal{O}}^{\mathrm{df}}$ is called the **degree filtered $\mathcal{O}$-border basis scheme**. Its affine coordinate ring is denoted by $B_{\mathcal{O}}^{\mathrm{df}} = K[c_{ij}]/I(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}})$.

Proposition 4.2 *For a* K*-rational point* $\Gamma = (\gamma_{ij})$ *of* $\mathbb{B}_{\mathcal{O}}$*, the 0-dimensional scheme* $\mathbb{X}_\Gamma$ *associated to* Γ *has a degree filtered* $\mathcal{O}$*-border basis if and only if* $\gamma_{ij} = 0$ *for all* $i \in \{1, \ldots, \mu\}$ *and* $j \in \{1, \ldots, \nu\}$ *such that* $\deg(t_i) > \deg(b_j)$*.*

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(b) The set of polynomials $G^{\mathrm{df}} = \{g_1^{\mathrm{df}}, \ldots, g_\nu^{\mathrm{df}}\}$ in $K[c_{ij}][x_1, \ldots, x_n]$ given by $g_j = b_j - \sum_{\{i \mid \deg(t_i) \leq \deg(b_j)\}} c_{ij}\, t_i$ is called the **generic degree filtered $\mathcal{O}$-border prebasis**.

Remark 4.4 (Some Properties of $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$)

Let C^{nondf} be the set of all indeterminates c_{ij} such that $\deg(t_i) > \deg(b_j)$.

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(a) For $k = 1, \ldots, n$, let $\mathcal{A}_k^{\mathrm{df}}$ be the matrix obtained from $\mathcal{A}_k$ by setting all indeterminates in C^{nondf} equal to zero. Then the matrices $\mathcal{A}_1^{\mathrm{df}}, \ldots, \mathcal{A}_n^{\mathrm{df}}$ are called the **generic degree filtered multiplication matrices** with respect to $\mathcal{O}$.

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Let C^{nondf} be the set of all indeterminates c_{ij} such that $\deg(t_i) > \deg(b_j)$.

(a) For $k = 1, \ldots, n$, let $\mathcal{A}_k^{\mathrm{df}}$ be the matrix obtained from $\mathcal{A}_k$ by setting all indeterminates in C^{nondf} equal to zero. Then the matrices $\mathcal{A}_1^{\mathrm{df}}, \ldots, \mathcal{A}_n^{\mathrm{df}}$ are called the **generic degree filtered multiplication matrices** with respect to $\mathcal{O}$.

(b) When we set the indeterminates in C^{nondf} equal to zero in $I(\mathbb{B}_{\mathcal{O}})$, we get an ideal $\bar{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}})$ such that $B_{\mathcal{O}}^{\mathrm{df}} \cong K[C^{\mathrm{df}}]/\bar{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}})$.

Remark 4.4 **(Some Properties of $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$)**
Let C^{nondf} be the set of all indeterminates c_{ij} such that $\deg(t_i) > \deg(b_j)$.

(a) For $k = 1, \ldots, n$, let $\mathcal{A}_k^{\mathrm{df}}$ be the matrix obtained from $\mathcal{A}_k$ by setting all indeterminates in C^{nondf} equal to zero. Then the matrices $\mathcal{A}_1^{\mathrm{df}}, \ldots, \mathcal{A}_n^{\mathrm{df}}$ are called the **generic degree filtered multiplication matrices** with respect to $\mathcal{O}$.

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(c) If $\mathcal{O}$ has a generic Hilbert function then $\mathbb{B}_{\mathcal{O}} = \mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$.

5 – Computing the Cayley-Bacharach Locus

Great! It is summer!

Finally I can wear short trousers

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Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$.

Definition 5.1 The set of all K-rational points $\Gamma = (\gamma_{ij}) \in K^{\mu\nu}$ of the border basis scheme $\mathbb{B}_\mathcal{O}$ whose associated 0-dimensional scheme $\mathbb{X}_\Gamma$ is a Cayley-Bacharach scheme is called the **Cayley-Bacharach locus** of $\mathbb{B}_\mathcal{O}$ and is denoted by $\mathrm{CB}(\mathcal{O})$.

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Goal: Calculate the Cayley-Bacharach locus in $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$, i.e. the equations defining $\mathrm{CB}(\mathcal{O}) \cap \mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$.

Algorithm 5.2 **(The Cayley-Bacharach Locus in $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$)**
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \#\{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\mathrm{NonCB}(\mathcal{O}) \cap \mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$.

Algorithm 5.2 (The Cayley-Bacharach Locus in $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$)
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(1) As above, calculate $I(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}) = I(\mathbb{B}_{\mathcal{O}}) + I_{\mathcal{O}}^{\mathrm{df}}$.

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(1) As above, calculate $I(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}) = I(\mathbb{B}_{\mathcal{O}}) + I_{\mathcal{O}}^{\mathrm{df}}$.

(2) Form the generic multiplication matrices $\mathcal{A}_1, \ldots, \mathcal{A}_n$. For $i = 1, \ldots, \mu$, compute the multiplication matrix $M_{t_i} = t_i(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

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(3) For $j = 1, \ldots, \Delta$, form the matrix $V_j \in \mathrm{Mat}_\mu(K[c_{ij}])$ whose i-th column is the $(\mu - \Delta + j)$-th column of $M_{t_i}^{\mathrm{tr}}$ for $i = 1, \ldots, \mu$.

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(5) Return the ideal $I(\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}) + J$.

6 – The Strict Complete Intersection Locus

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Definition 6.1 Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{A}^n$. The scheme $\mathbb{X}$ is called a **strict complete intersection scheme** if the associated graded ring $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}}) \cong P/\operatorname{DF}(I)$ is a (local) complete intersection.

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Idea: The rings $P/\,\mathrm{DF}(I)$ are parametrized by the **homogeneous border basis scheme**. Apply the characterization of local complete intersections to this family.

Definition 6.2 Let $I_{\mathcal{O}}^{\mathrm{hom}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates c_{ij} such that $\deg(t_i) \neq \deg(b_j)$.

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(b) The set of polynomials $G^{\text{hom}} = \{g_1^{\text{hom}}, \ldots, g_\nu^{\text{hom}}\}$ in $K[c_{ij}][x_1, \ldots, x_n]$ given by $g_j^{\text{hom}} = b_j - \sum_{\{i \mid \deg(t_i) = \deg(b_j)\}} c_{ij}\, t_i$ is called the **generic homogeneous $\mathcal{O}$-border prebasis**.

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(c) Let C^{hom} be the set of all c_{ij} such that $\deg(t_i) \neq \deg(b_j)$. For $k = 1, \ldots, n$, let $\mathcal{A}_k^{\mathrm{hom}}$ be the matrix obtained from $\mathcal{A}_k$ by setting all indeterminates in C^{hom} equal to zero. Then the matrices $\mathcal{A}_1^{\mathrm{hom}}, \ldots, \mathcal{A}_n^{\mathrm{hom}}$ are called the **generic homogeneous multiplication matrices** with respect to $\mathcal{O}$.

Algorithm 6.3 **(Computing the Strict CI Locus in $\mathbb{B}_{\mathcal{O}}^{\mathrm{df}}$)**
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\varrho = \deg(t_\mu)$. Consider the following sequence of instructions.

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(1) For $i = 1, \dots, \varrho$, determine the number $h_i = \#\{t_j \in \mathcal{O} \mid \deg(t_j) = i\}$. If the tuple $(h_0, \dots, h_\varrho)$ is not symmetric, then return the ideal $\langle 1 \rangle$ and stop.

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(3) Form the generic homogeneous $\mathcal{O}$-border prebasis $G^{\mathrm{hom}} = \{g_1^{\mathrm{hom}}, \ldots, g_j^{\mathrm{hom}}\}$ and write $g_j^{\mathrm{hom}} = \sum_{i=1}^{n} h_{ij} x_i$ with $h_{ij} \in K[c_{ij}][x_1, \ldots, x_n]$ for $j = 1, \ldots, \nu$.

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(4) Form the matrix W of size $n \times \nu$ whose columns are given by $\sum_{i=1}^{n} h_{ij} e_i$ for $j = 1, \ldots, \nu$.

(5) Let $k = \binom{\nu}{n}$. Calculate the minors $f_1, \ldots, f_k$ of order n of W.

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(6) Using border division by $G^{\rm hom}$, write the residue classes $\bar{f}_1, \ldots, \bar{f}_k \in B_{\mathcal{O}}^{\rm hom}/\langle G^{\rm hom} \rangle$ as $B_{\mathcal{O}}^{\rm hom}$-linear combinations $\bar{f}_j = \sum_{i=1}^{\mu} \bar{a}_{ij} t_i$ with $\bar{a}_{1j}, \ldots, \bar{a}_{\mu j} \in B_{\mathcal{O}}^{\rm hom}$ for $j = 1, \ldots, k$.

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(7) Let $C^{\rm hom} = \{c_{ij} \mid \deg(t_i) = \deg(b_j)\}$. For $i = 1, \dots, \mu$ and $j = 1, \dots, k$, let $a_{ij} \in K[C^{\rm hom}]$ be a polynomial which represents the $\bar{a}_{ij}$ with respect to $B_{\mathcal{O}}^{\rm hom} \cong K[C^{\rm hom}]/\bar{I}(\mathbb{B}_{\mathcal{O}}^{\rm hom})$. Return the ideal $J = I(\mathbb{B}_{\mathcal{O}}^{\rm df}) + \langle a_{ij} \mid i \in \{1, \dots, \mu\}, j \in \{1, \dots, k\}\rangle$ and stop.

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This is an algorithm which computes an ideal J in the ring $K[c_{ij}]$ which defines a closed subscheme $\mathrm{NonSCI}(\mathcal{O}) \cap \mathbb{B}_{\mathcal{O}}^{\rm df}$. The K-rational points of this subscheme represent the 0-dimensional subschemes of $\mathbb{A}^n$ which have a degree filtered $\mathcal{O}$-border basis, but are **not strict complete intersection schemes**.

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• The closed subscheme $\mathbb{B}_{\mathcal{O}}(\overline{\mathcal{H}})$ of $\mathbb{B}_{\mathcal{O}}$ corresponds to all schemes whose Hilbert function is dominated by a fixed Hilbert function $\mathcal{H}$.

• Its open subset $\mathbb{B}_{\mathcal{O}}(\mathcal{H})$ corresponds to all schemes whose Hilbert function is $\mathcal{H}$.

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(4) Inside the parts of this **stratificaton** we can calculate the equations defining the loci of the subschemes which are locally Gorenstein, Cayley-Bacharach, strict complete intersections, etc. In general, these loci are **constructible** and can be described by a pair of ideals.

THE END

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Humor is if you laugh anyway.