Fast Gröbner basis computation and polynomial reduction for generic bivariate ideals

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ACA 2018 – Santiago de Compostela, Spain
Session "Algorithms for Zero-Dimensional Ideals"
19 / 06 / 2018

Introduction

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Setting and notations

- $I = \langle A, B \rangle$ with generic $A, B \in \mathbb{K}[X, Y]$ given in total degree.
- Use the degree lexicographic order to compute G.
- $\deg A = n$ and $\deg B = m$ with $n \leqslant m$ (in this talk $n = m$)
- We want to reduce P with $\deg P = d$

Main result

In this specific setting, a quasi-optimal algorithm exists !

Outline

1. Presentation of the problem
   - Polynomial reduction: complexity
   - Gröbner bases: concise representation
2. Faster computation
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   - Gröbner basis

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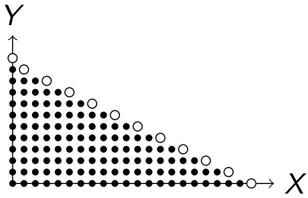
2. Faster computation

Polynomial reduction: complexity



- A, B: $O(n^2)$ coefficients
- $\mathbb{K}[X, Y]/I$: dimension $O(n^2)$
- G: $O(n^3)$ coefficients ($O(n^2)$ for each G_i)

Reduction using G needs at least $O(n^3)$ $\Longrightarrow$ reduction with less information?

Related result

Theorem (van der Hoeven, L. – ISSAC 2018)

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- Problem: in this setting, G is *not* vanilla.
- But . . . similar ideas still apply.

Gröbner bases: concise representation – 1

The Gröbner basis is generated by A and $B \implies$ there are relations between the G_i (redundant information)

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$G_0 \cong A$, $G_1 \cong B$ and well-chosen $M_{i,k}$ hold all information about G. Also, little information is required to compute the $M_{i,k}$.

Gröbner bases: concise representation – 2

The coefficients of each G_i are needed to compute the reduction, but there are too many.

- Keep only enough coefficients to evaluate Q_i
- Then, rewrite $G_i = f(G_k, G_{k+1})$ to evaluate the remainder.

Gröbner bases: concise representation – 2

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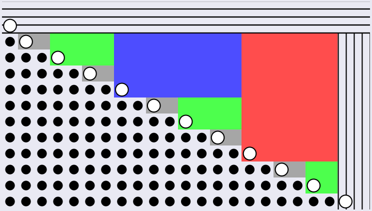
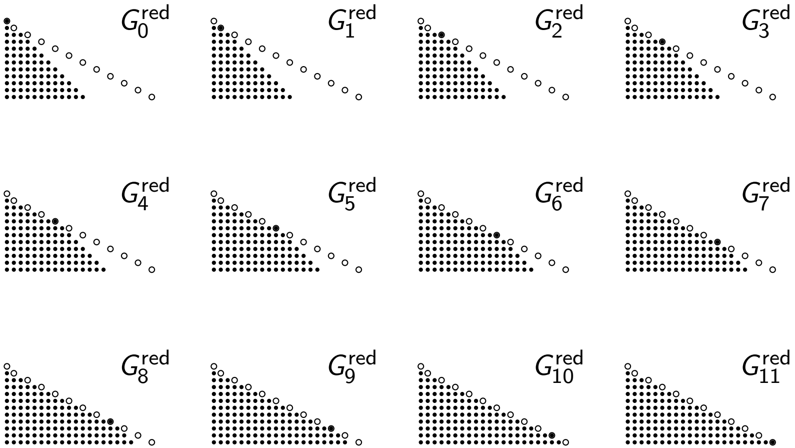
$\Longrightarrow$ Control the degree of the quotients.

Dichotomic selection strategy



- $n/2$ quotients of degree 1
- $n/4$ quotients of degree 4
- $n/8$ quotients of degree 10
- ...
- $n/2^i$ quotients of degree $3 \times 2^{i-1} - 2$

Gröbner bases: concise representation – Example



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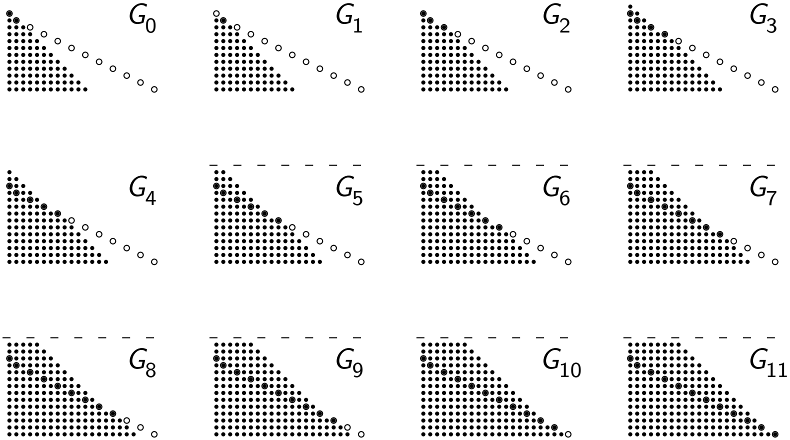

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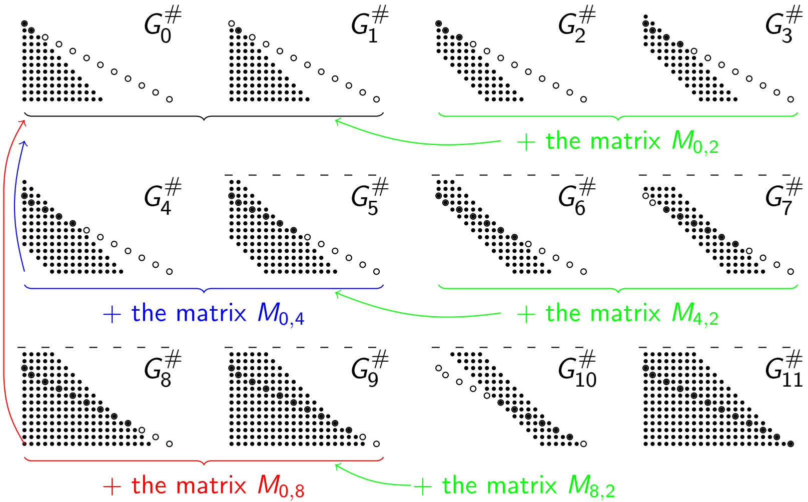

Outline

Polynomial reduction – Overview

Theorem (van der Hoeven – ACA 2015)

Using relaxed multiplications, the extended reduction of P modulo G can be computed in quasi-linear time for the size of the equation

$$P = \sum_i Q_i G_i + R$$

But this equation has size $O(n^3)$ and we would like to achieve $\tilde{O}(n^2)$.

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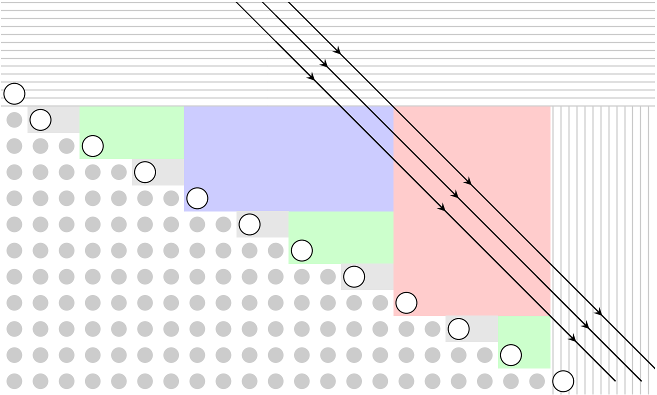
Adapt the algorithm to take advantage of the concise representation:

- Use the relaxed algorithm to compute the quotients.
- Once Q_i is known, replace $Q_i G_i$ by $S_k G_k + S_{k+1} G_{k+1}$ to increase precision.

Polynomial reduction – Example



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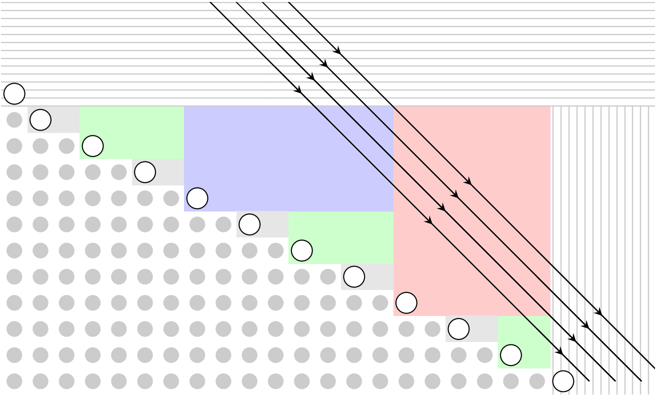

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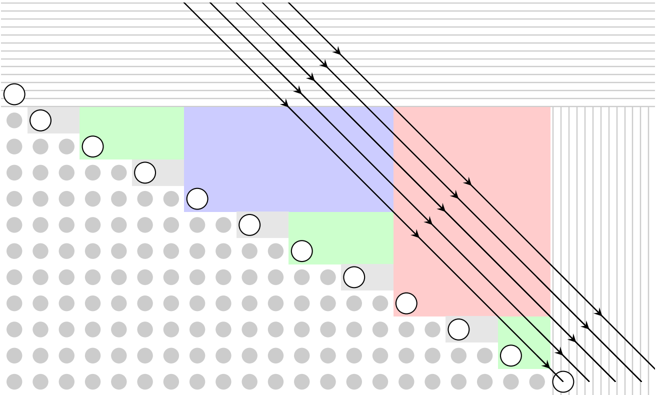

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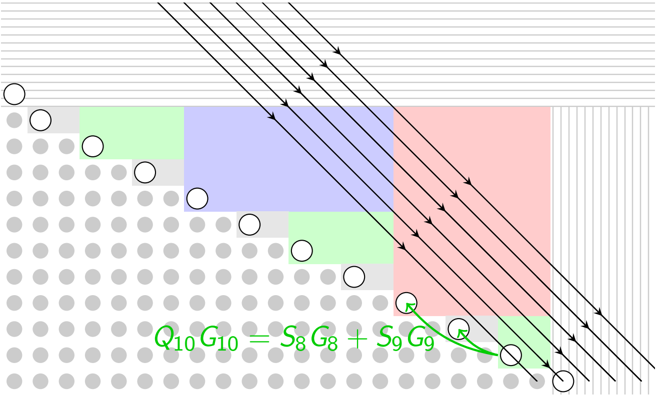

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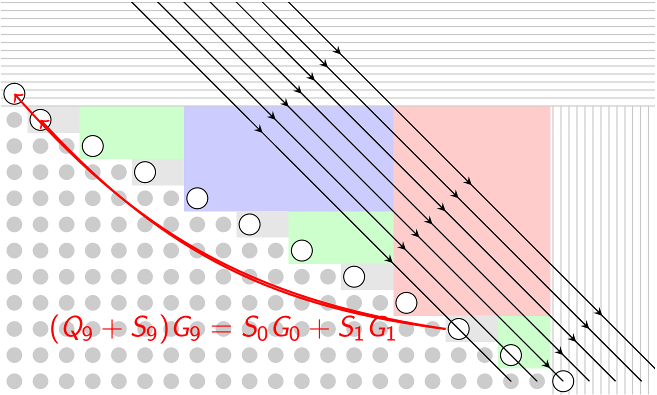

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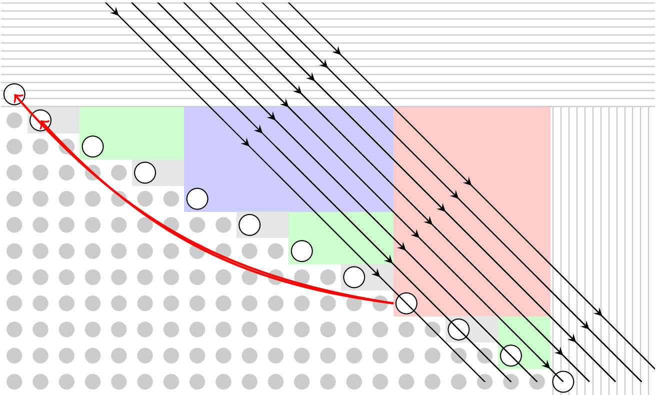

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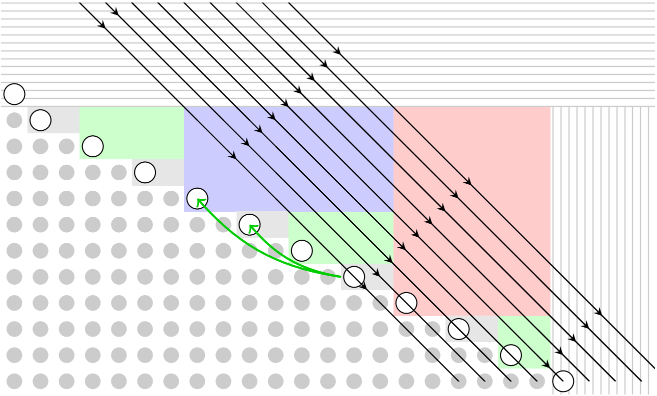

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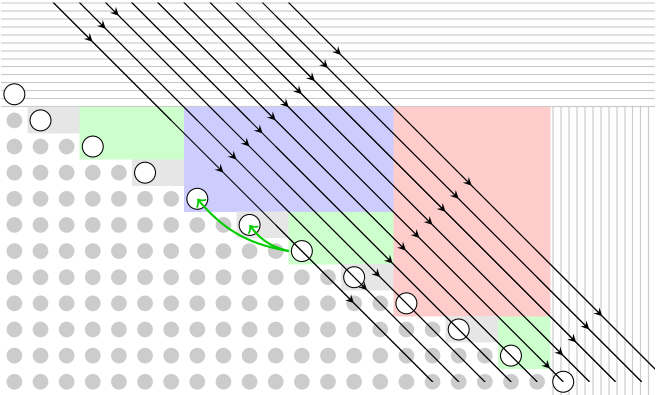

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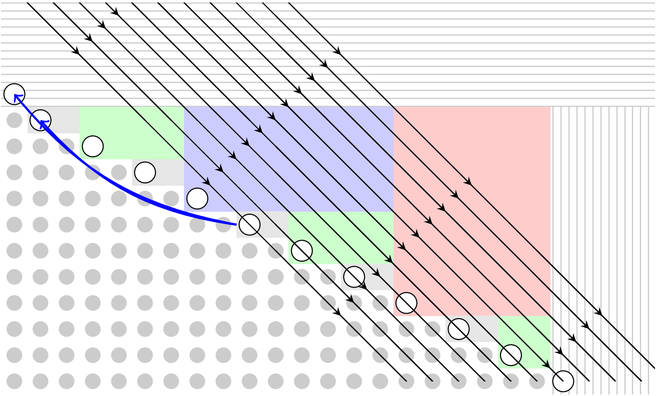

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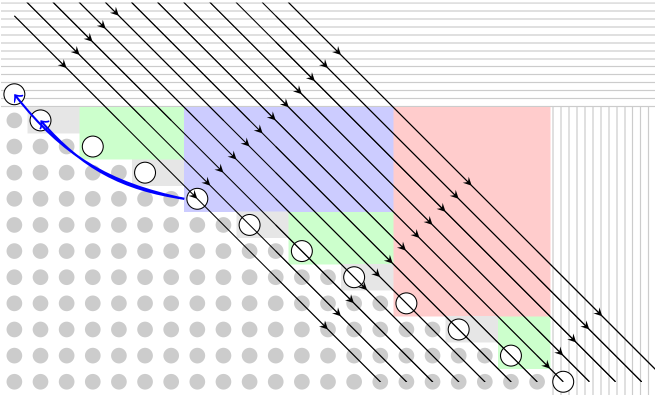

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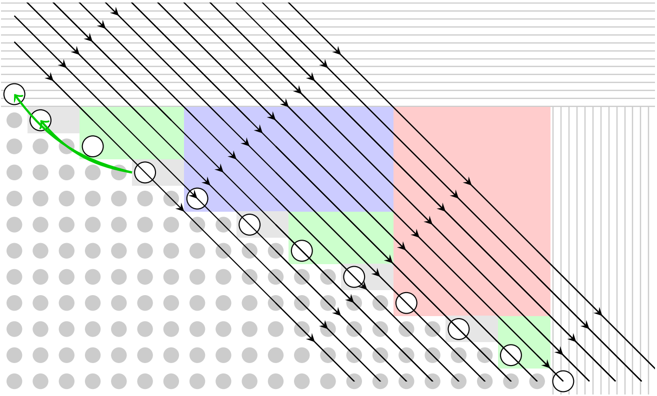

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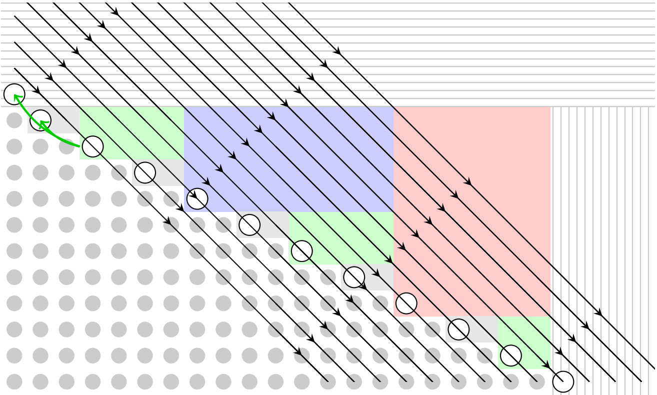

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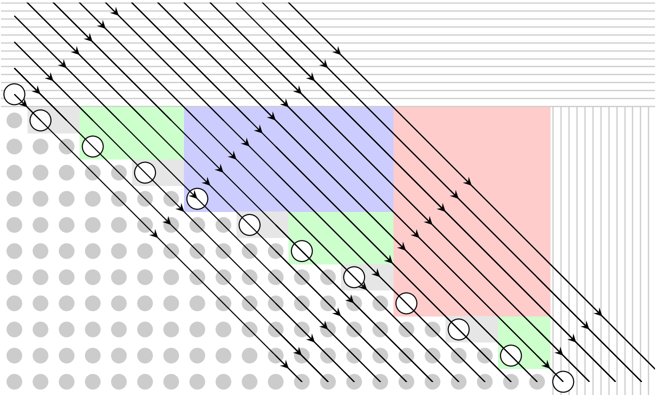

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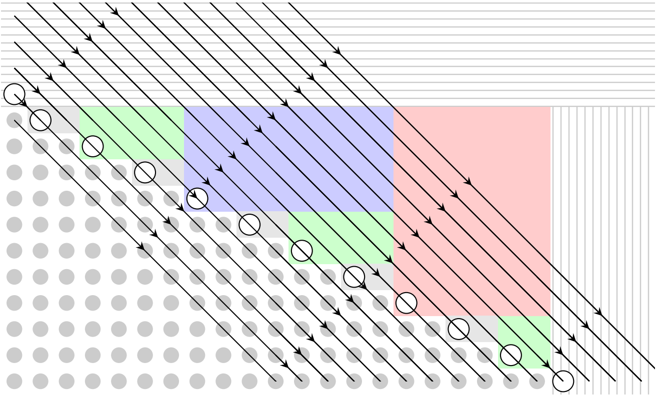

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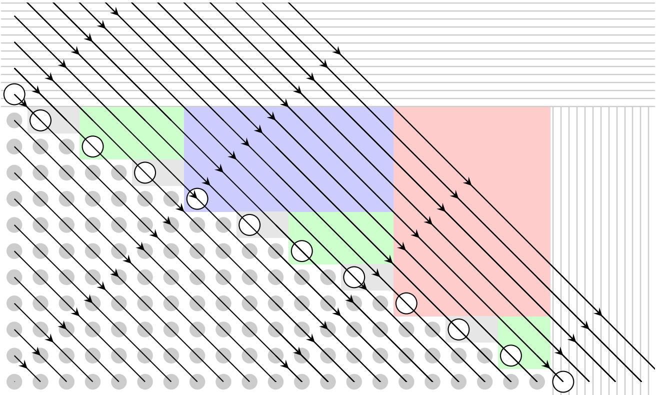

Gröbner basis

- Compute the concise representation: $\tilde{O}(n^2)$
- Compute the non-reduced Gröbner basis using the simple recurrence relations: $\tilde{O}(n^2)$ for each element $\Longrightarrow$ $\tilde{O}(n^3)$
- Reduce each basis element using the fast algorithm: $\tilde{O}(n^2)$ for each element $\Longrightarrow$ $\tilde{O}(n^3)$

Conclusion

Main result

In a generic bivariate setting, there are quasi-optimal algorithms for polynomial reduction (in terms of the size of A, B, P) and to compute the reduced Gröbner basis (in terms of the output size)

Generalization:

- Slightly degenerate cases ?
- More than 2 variables ?

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Generalization:

- Slightly degenerate cases ? $\rightarrow$ seems feasible.
- More than 2 variables ? $\rightarrow$ much more difficult.

Conclusion

Proof-of-concept implementation (in Sage) at
`https://www.lix.polytechnique.fr/~larrieu/`

- Mainly intended as correctness proof.
- Missing (fast) implementation of some primitives $\Longrightarrow$ reduction is not competitive in practice.
- Computing the concise representation is faster than Sage's builtin Gröbner basis library for $n \geqslant 160$.

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Thank you for your attention