De Nugis Groebnerialium 5: Noether, Macaulay, Jordan

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Re-reading Macaulay

References

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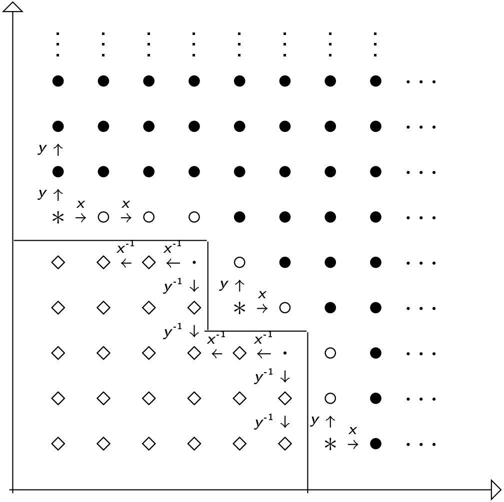
Inverse system

$$\ell = \sum_{\tau \in \mathcal{T}} c_\tau \tau^{-1} \in k[[X_1^{-1}, \ldots, X_n^{-1}]];\ f = \sum_{t \in \mathcal{T}} a_t t \in k[X_1, \ldots, X_n]$$

$$s := \sum_{\tau \in \mathcal{T}} \sum_{t \in \mathcal{T}} a_t c_\tau \tau^{-1} t \in k((X_1, \ldots, X_n))$$

$$\ell(f) = s(0, \ldots, 0) = \sum_{t \in \mathcal{T}} a_t c_t$$

ℓ **inverse system** if $\sum_{t \in \mathcal{T}} a_t c_t = 0,\ \forall f = \sum_{t \in \mathcal{T}} a_t t \in \mathrm{I}$.



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For each $\tau \in \mathcal{T}$, denote $M(\tau) : \mathcal{P} \to k$ the morphism defined by

$$M(\tau) = c(f, \tau), \forall f = \sum_{t \in \mathcal{T}} c(f, t) t \in \mathcal{P}.$$

Gröbner gave a natural description of each functional $M(\tau) \in \mathbb{M}$ in terms of differential operations, setting, for each $(i_1, \ldots, i_n) \in \mathbb{N}^n$, $\tau := x_1^{i_1} \ldots x_n^{i_n}$ and denoting

$$D(\tau) := D(i_1, \ldots, i_n) : \mathcal{P} \to \mathcal{P}$$

the differential operator $D(\tau) := D(i_1, \ldots, i_n) = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$, so that, $\forall \tau \in \mathcal{P}$, it holds $M(\tau)(\cdot) = D(\tau)(\cdot)(0, \ldots, 0)$.
Gröbner's formulation has the only weakness of requiring $\mathrm{char}(\mathbf{k}) = 0$, but this problem is fixed using the **Hasse derivates** $D_i^{(j)}(x_i^m) = \begin{cases} \binom{m}{j} x^{m-j} & \text{if } m \geq j \\ 0 & \text{if } m < j \end{cases}$ thus obtaining

$$M(\tau)(\cdot) = D_1^{(i_1)} \cdots D_n^{(i_n)}(\cdot)(0, \ldots, 0)$$

A stability matter

$$\forall \tau \in \mathcal{T}, X_i \cdot M(\tau) = \begin{cases} M(\frac{\tau}{X_i}) & \text{if } X_i \mid \tau \\ 0 & \text{if } X_i \nmid \tau \end{cases}$$

A k-vector subspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ is called **stable** if

$$\lambda \in \Lambda \implies X_i \cdot \lambda \in \Lambda$$

i.e. Λ is a $\mathcal{P}$-module.

Clearly $\mathcal{P}^* \cong k[[\mathbb{M}]]$; however in order to have reasonable duality we must restrict ourselves to $\mathrm{Span}_k(\mathbb{M}) \cong k[\mathbb{M}]$.
For each k-vector subspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ denote

$$\mathfrak{I}(\Lambda) := \mathfrak{P}(\Lambda) = \{f \in \mathcal{P} : \ell(f) = 0, \forall \ell \in \Lambda\}$$

and for each k-vector subspace $P \subset \mathcal{P}$ denote

$$\begin{aligned} \mathfrak{M}(P) &:= \mathfrak{L}(P) \cap \mathrm{Span}_K(\mathbb{M}) \\ &= \{\ell \in \mathrm{Span}_K(\mathbb{M}) : \ell(f) = 0, \forall f \in P\}. \end{aligned}$$

The mutually inverse maps $\mathfrak{I}(\cdot)$ and $\mathfrak{M}(\cdot)$ give a *biunivocal, inclusion reversing, correspondence* between the set of the *m-closed ideals* $\mathsf{I} \subset \mathcal{P}$ and the set of the *stable k-vector subspaces* $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$.

They are the restriction of, respectively, $\mathfrak{P}(\cdot)$ to m-closed ideals $I \subset \mathcal{P}$, and $\mathfrak{L}(\cdot)$ to stable k-vector subspaces $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$. Moreover, for any m-primary ideal $\mathfrak{q} \subset \mathcal{P}$, $\mathfrak{M}(\mathfrak{q})$ is finite k-dimensional and we have

$$\deg(\mathfrak{q}) = \dim_K(\mathfrak{M}(\mathfrak{q}));$$

conversely for any finite k-dim. stable k-vector subspace $\Lambda \subset \mathrm{Span}_K(\mathbb{M})$, $\mathfrak{I}(\Lambda)$ is an m-primary ideal and we have

$$\dim_k(\Lambda) = \deg(\mathfrak{I}(\Lambda)).$$

Let $<$ be a semigroup ordering on $\mathcal{T}$ and $\mathsf{I} \subset \mathcal{P}$ an m-closed ideal.

$$\mathrm{Can}(t, \mathsf{I}, <) =: \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in k[[\mathbf{N}_<(\mathsf{I})]] \subset k[[X_1, \ldots, X_n]]$$

so that

$$t - \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in \mathsf{I},$$
$$t < \tau \implies \gamma(t, \tau, <) = 0.$$

Define, for each $\tau \in \mathbf{N}_<(\mathsf{I})$,

$$\ell(\tau) := M(\tau) + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <)M(t) \in k[[\mathbb{M}]].$$

$$\ell(\tau) := \tau^{-1} + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <)t^{-1} \in k[[X_1^{-1}..X_n^{-1}]].$$

Remark that $\ell(\tau) \in \mathfrak{M}(\mathsf{I})$ requires $\ell(\tau) \in k[\mathbb{M}]$ which holds iff $\{t : \gamma(t, \tau, <) \neq 0\}$ is finite and is granted if $\{t : t > \tau\}$ is finite. To obtain this we must choose as $<$ a *standard ordering* i.e. s.t.:

- $X_i < 1, \forall i$,
- for each infinite decreasing sequence in $\mathcal{T}$

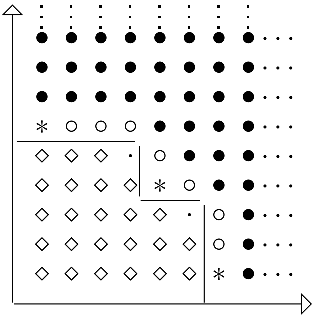
$$\tau_1 > \tau_2 > \cdots \tau_\nu > \cdots$$

  and each $\tau \in \mathcal{T}$ there is $\nu : \tau > \tau_\nu$.

In this setting the generalization of the notion of Gröbner basis is called **Hironoka/standard basis** and deals with *series* instead of polynomials.
The choice of this setting is natural, since a Hironaka basis of an ideal I returns its m-closure.

$\mathcal{P} := k[X_1, \ldots, X_n]$, $\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\}$, $\prec$ term-order on $\mathcal{T}$, $f = \sum_{\tau \in \mathcal{T}} c(f, \tau)\tau \in \mathrm{Span}_k(\mathcal{T}) = \mathcal{P}$, $\mathbf{T}(f) := \max_{<}\{\tau \in \mathcal{T} : c(f, \tau) \neq 0\}$, $\mathsf{I} \subset \mathcal{P}$ a (0)-dim. ideal,

$\mathbf{T}(\mathsf{I}) := \{\mathbf{T}(f) : f \in \mathsf{I}\}$ monomial ideal,
$\diamond$ $\mathbf{N}(\mathsf{I}) := \mathbf{N}_{<}(\mathsf{I}) = \mathcal{T} \setminus \mathbf{T}_{<}(\mathsf{I})$ order ideal,
$\circ$ $\mathbf{B}_{\prec}(\mathsf{I}) := \{X_h\tau : 1 \leq h \leq n, \tau \in \mathbf{N}_{\prec}(\mathsf{I})\} \setminus \mathbf{N}_{\prec}(\mathsf{I})$,
$\bullet$ $\mathbf{I}_{\prec}(\mathsf{I}) := \mathbf{T}_{\prec}(\mathsf{I}) \setminus \mathbf{B}_{\prec}(\mathsf{I})$,
$*$ $\mathbf{G}_{\prec}(\mathsf{I}) \subset \mathbf{B}_{\prec}(\mathsf{I})$ the unique minimal basis of $\mathbf{T}_{\prec}(\mathsf{I})$,
$\cdot$ $\mathbf{C}_{\prec}(\mathsf{I}) := \{\tau \in \mathbf{N}_{\prec}(\mathsf{I}) : X_h\tau \in \mathbf{T}_{\prec}(\mathsf{I}), \forall h\}$.



$\mathsf{m} = (X_1, \dots, X_n) \subset \mathcal{P}$,

$<$ standard-ordering on $\mathcal{T}$,

I m-closed ideal,

$\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \dots, \omega_s\}$, finite corner set

the (not-necessarily finite) set $\mathbf{N}_<(\mathsf{I})$,

the Macaulay basis $\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\}$,

$\Lambda := \mathrm{Span}_k\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\} \subset \mathrm{Span}_k(\mathbb{M})$;

$$\forall j, 1 \leq j \leq s, \Lambda_j := \mathrm{Span}_k\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\}$$

$$\forall j, 1 \leq j \leq s, \mathfrak{q}_j := \mathfrak{I}(\Lambda_j)$$

Lemma (Macaulay)

With the notation above, for each j, denoting

$$\Lambda'_j := \mathrm{Span}_K\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T} \cap \mathsf{m}\}$$

we have

$$\dim_K(\Lambda'_j) = \dim_K(\Lambda_j) - 1,$$
$$\ell(\omega_j) \notin \Lambda'_j = \mathfrak{M}(\mathfrak{q}_j : \mathsf{m}),$$
$$\mathfrak{q}' \supset \mathfrak{q}_j \implies \mathfrak{M}(\mathfrak{q}') \subseteq \Lambda'_j.$$

Macaulay Chain

$\ell \in \mathrm{Span}_K(\mathbb{M})$, $[\ell] = \mathfrak{I}((\mathrm{Span}_K\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\})$
An ideal I, $\dim(\mathsf{I}) > 0$, is called a **principal system** if there is a chain of zero-dimensional principal systems $\mathsf{I}_i := [E_i]$ such that

$$\mathsf{I} = \cap_i \mathsf{I}_i \text{ and}$$

$$\mathsf{I}_1 \supset \mathsf{I}_2 \supset \cdots \mathsf{I}_i \supset \mathsf{I}_{i+1} \supset \cdots \supset \mathsf{I}$$

Lemma (Macaulay)

Let q *be a primary at the origin,* $\deg(q) = \mu$*. Then there is an ordered set of inverse functions* $\{e_1, \ldots, e_\mu\}$ *such that*

- $q = [e_1, \ldots, e_\mu]$,
- *for each* $i \leq \mu$,
  - $\mathrm{Span}_k(\{e_1, \ldots, e_i\})$ *is closed under derivation,*
  - $\dim_k(\mathrm{Span}_k(\{e_1, \ldots, e_i\})) = i$.

Corollary

Let $\mathfrak{q}$ *be a primary at the origin,* $\deg(\mathfrak{q}) = \mu$*.*
Let $\{e_1, \ldots, e_\mu\}$ *be any ordered set of inverse functions satisfying the properties above and, for each* i *define* $\mathfrak{q}_i := [e_1, \ldots, e_i]$*. Then*

- $\mathfrak{q}_i$ *is a primary ideal at the origin, for each* i*;*
- $\deg(\mathfrak{q}_i) = i$*, for each* i*;*
- $\mathfrak{p} = \mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_{\mu-1} \supset \mathfrak{q}_\mu = \mathfrak{q}$.

Let $J \subset \{1, \ldots, s\}$ be the set such that $\{\mathfrak{q}_j : j \in J\}$ is the set of the minimal elements of $\{\mathfrak{q}_j : 1 \leq j \leq s\}$ and remark that $\mathfrak{q}_i \subset \mathfrak{q}_j \iff \Lambda_i \supset \Lambda_j$.

Theorem (Groebner)

If I *is* m*-primary, then:*

1. *each* Λ_j *is a finite-dim. stable vector space;*
2. *each* $\mathfrak{q}_j$ *is an* m*-primary ideal,*
3. *is **reduced***
4. *and irreducible.*
5. $\mathsf{I} := \cap_{j \in J} \mathfrak{q}_j$ *is a **reduced representation** of* I.

In connection with Lasker-Noether primary decomposition, Emmy Noether stated that

Definition (Noether)

Let R be a commutative ring with unity and let $\mathfrak{a} \subset R$ be an ideal. $\mathfrak{a}$ is said to be

- *reducible* if there are two ideals $\mathfrak{b}, \mathfrak{c} \subset R$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}, \mathfrak{b} \supset \mathfrak{a}, \mathfrak{c} \supset \mathfrak{a}$;
- *irreducible* if it is not reducible.

Proposition (Lasker–Noether)

In a Noetherian ring R each ideal $\mathfrak{f} \subset R$ is a finite intersection of irreducible ideals:

$$\mathfrak{f} = \cap_{i=1}^{r} \mathfrak{i}_i .$$

Definition (Noether)

Let R be a Noetherian ring and $\mathfrak{f} \subset R$ an ideal. A representation $\mathfrak{f} = \cap_{i=1}^{r} \mathfrak{i}_i$, of $\mathfrak{f}$ as intersection of finite irreducible ideals is called a **reduced representation** if, for each $l, 1 \leq l \leq r$,

- $\mathfrak{i}_l \not\supseteq \bigcap_{\substack{i=1 \\ i \neq l}}^{r} \mathfrak{i}_i$, and
- there is no irreducible ideal $\mathfrak{i}'_l \supset \mathfrak{i}_l$ such that

$$\mathfrak{f} = \left(\bigcap_{\substack{i=1 \\ i \neq l}}^{r} \mathfrak{i}_i \right) \bigcap \mathfrak{i}'_l .$$

Proposition (Noether)

In a Noetherian ring R, each ideal $\mathfrak{f} \subset R$ has a *reduced representation* as intersection of finite *irreducible ideals*.

The decomposition

$$(X^2, XY) = (X) \cap (X^2, XY, Y^\lambda), \forall \lambda \in \mathbb{N}, \lambda \geq 1,$$

where $\sqrt{(X^2, XY, Y^\lambda)} = (X, Y) \supset (X)$, shows that *embedded components are not unique*; however,

$$(X^2, XY, Y) = (X^2, Y) \supseteq (X^2, XY, Y^\lambda), \forall \lambda > 1,$$

shows that (X^2, Y) is a **reduced embedded irreducible** component and that

$$(X^2, XY) = (X) \cap (X^2, Y)$$

is a **reduced representation**.
The decompositions

$$(X^2, XY) = (X) \cap (X^2, Y + aX), \forall a \in \mathbb{Q},$$

where $\sqrt{(X^2, Y + aX)} = (X, Y) \supset (X)$ and, clearly, each $(X^2, Y + aX)$ is reduced, show that also *reduced representations are not unique*; remark that, setting $a = 0$, we find again the previous one $(X^2, XY) = (X) \cap (X^2, Y)$.

If I is not m-primary, let

$\rho := \max\{\deg(\omega_j) + 1 : \omega_j \in \mathbf{C}(I)\}$ so that

$\mathfrak{q}' := I + m^\rho$ is an m-primary component of I;

$I = \cap_{i=1}^r \mathfrak{q}_i$ an irredundant primary representation of I with $\sqrt{\mathfrak{q}_1} = m$;

$\mathfrak{b} := I : \mathfrak{m}^\infty = \cap_{i=2}^r \mathfrak{q}_i$;

$\mathfrak{b} = \cap_{i=1}^u \mathfrak{Q}_i$, a reduced representation of $\mathfrak{b}$;

$\mathfrak{q}_1 := \cap_{j=1}^s \mathfrak{q}_j$ a reduced representation of $\mathfrak{q}_1$ which is wlog ordered so that $\mathfrak{q}_i \supset \mathfrak{b} \iff i > t$; $\mathfrak{q} := \cap_{j=1}^t \mathfrak{q}_j$.

Then

1. $\mathfrak{q}$ is a *reduced m-primary component* of I,
2. $\mathfrak{q} := \cap_{j=1}^t \mathfrak{q}_j$ is a *reduced representation* of $\mathfrak{q}$,
3. $I = \cap_{i=1}^u \mathfrak{Q}_i \bigcap \cap_{j=1}^t \mathfrak{q}_j$ is a *reduced representation* of I.

Example

$\mathsf{I} := (X^2, XY)$, $\Lambda = \mathrm{Span}_k\{M(1), M(X)\} \cup \{M(Y^i), i \in \mathbb{N}\}$;

$\rho = 2$, $\mathfrak{M}(\mathsf{I} + \mathsf{m}^2) = \{M(1), M(X), M(Y)\}$,

$\omega_1 := X, \Lambda_1 = \{M(1), M(X))\}, \mathfrak{q}_1 = (X^2, Y)$,

$\omega_2 := Y, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^2)$,

$\mathsf{I} : \mathsf{m}^\infty = (X) \subset (X, Y^2)$,

$$(X^2, XY) = (X) \cap (X^2, Y)$$

Both the **reduced representation** and the notion of **Macaulay basis** strongly depend on the choice of a **frame of coordinates**. In fact, considering, for each $a \in \mathbb{Q}, a \neq 0$,

$$\Lambda = \mathrm{Span}_k\{M(1), M(X) - aM(Y)\} \cup \{M(Y^i), i \in \mathbb{N}\},$$

we obtain

$$\begin{aligned}
&\rho = 2, \\
&\mathfrak{M}(\mathsf{I} + \mathsf{m}^2) = \{M(1), M(X) - aM(Y), M(Y)\}, \\
&\omega_1 := X, \Lambda_1 = \{M(1), M(X) - aM(Y)\}, \mathfrak{q}_1 = (X^2, Y + aX), \\
&\omega_2 := Y, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^2), \\
&\mathsf{I} : \mathsf{m}^\infty = (X) \subset (X, Y^2), \\
&(X^2, XY) = (X) \cap (X^2, Y + aX).
\end{aligned}$$

Question

Macaulay's algorithm effectively computes an irredundant (and reduced) representation as finite intersection of irreducible primary ideals. Moreover, once a frame of coordinates is fixed, such decomposition is unique.

Could this result allow to define (if and when it exists) an intrinsic coordinate frame for primary ideals?

Apparently, the previous example is all one needs to dismiss this hope; however if we consider any linear form $\ell \in K[X_1, X_2, X_3]$ s.t. $\mathrm{Span}_K = \{X_1, X_2, \ell\} = \mathrm{Span}_K = \{X_1, X_2, X_3\}$ we realize that in the (X_1, X_2, X_3)-primary ideal

$$
\begin{aligned}
J &:= (X_1, X_2, X_3)^2 \cap (X_1, X_2, \ell^3) \\
&= (X_1^2, X_1X_2, X_2^2, X_1X_3, X_2X_3, X_3^3) \\
&= ((aX + bY)^2, cX + dY, X_3) \cap (aX + bY, (cX + dY)^2, X_3) \cap (X_1, X_2, \ell^3)
\end{aligned}
$$

the coordinate X_3 plays a rôle at least as the direction of the plane (X_1, X_2).

Let us consider a $(X_1, \dots, X_n)$-primary ideal $I \subset \mathcal{P}$, the unique order ideal $\mathbf{N}(I) \subset \mathcal{T}$ such that $\mathrm{Span}_K\{\mathbf{N}(I)\} = \mathcal{P}/I$, a linear form

$$\ell \in \mathrm{Span}_K\{X_1, \cdots, X_n\} =: \mathcal{B}_1,$$

the Auzinger-Stetter matrix A describing the effect of the morphism $A \to A : f \mapsto \ell f$ *on* $\mathbf{N}(I)$ and its Jordan normal form J. Denoting, for $k, 1 \leq k \leq \#\mathbf{N}(J)$, $\rho_k := \mathrm{rank}(A^{k-1}) - \mathrm{rank}(A^k)$, $\mu_0 := \rho_1$ and $\mu_i := \rho_i - \rho_{i+1}$ for each $i, 1 \leq i < l := max(k : \rho_k \neq 0)$. Note that $\mu_0 = \sum_{i>0} \mu_i = \#\mathcal{B}_1 = n$ is the number of Jordan blocks of J. Note also that the following conditions are equivalent

1. there are n values $i_1 > i_2 > \dots > i_n$ with $\mu_{i_j} = 1$,
2. $\mu_i \in \{0, 1\}$ for each i.

If this happens we can choose n generalized eigenvectors v_j each of ranks i_j in a such way that the eigenvectors $w_j := A^{i_j-1}v_j$ satisfy $\mathrm{Span}_K\{w_1, \cdots, w_n\} =: \mathcal{B}_1$ and we can inductively choose each w_j in such a way that the basis $\{w_1, \cdots, w_n\}$ is orthogonal.

Definition

If the conditions above are satisfied the ordered set $\{w_1, \cdots, w_n\}$ is called the **intrinsic coordinate frame** for the $(X_1, \ldots, X_n)$-primary ideal I.

Continuation

For each $j \in \{1, \ldots, n\}$, $\sigma_j, \rho_j, \lambda_j \in End_k(\mathrm{Span}_k(\mathbb{M}))$ are defined as follows:

$$\sigma_j(M(\tau)) := \sigma_{X_j}(M(\tau)) = \begin{cases} M(\omega) \text{ if } \tau = X_j\omega \\ 0 \text{ if } X_j \nmid \tau \end{cases} \qquad \forall\ \tau \in \mathcal{T};$$

$$\rho_j(M(\tau)) := \rho_{X_j}(M(\tau)) = M(X_j\tau) \quad \forall\ \tau \in \mathcal{T};$$

$$\lambda_j(M(\tau)) = \begin{cases} M(\tau) \text{ if } X_j \mid \tau \\ 0 \text{ if } X_j \nmid \tau \end{cases} \qquad \forall\ \tau \in \mathcal{T}.$$

Let $<$ be an inf-limited ordering, $\mathsf{I} \subset \mathcal{Q}$ an m-primary ideal, $V := \mathfrak{M}(\mathsf{I})$, $\Lambda := \{\ell_1, \dots, \ell_s\}$ be a Macaulay basis of V.
Any element

$$\ell := M(\mathbf{T}_<(\ell)) + \sum_{\omega \in \mathcal{W}} c_\omega M(\omega) \in \mathrm{Span}_K(\mathbb{M}) \setminus V$$

such that

c1) $\mathbf{T}_<(\ell) \in \mathbf{C}_<(V)$,

c2) $\sigma_j(\ell) \in V$ for each j,

c3) $c_\omega \neq 0 \implies \omega \notin \mathbf{T}_<\{V\}$.

is called a **continuation** of V at $\tau := \mathbf{T}_<(\ell)$.

An **elementary** continuation of V at $\tau \in \mathbf{C}_<(V)$ is a continuation

$$\ell := M(\mathbf{T}_<(\ell)) + \sum_{\omega \in \mathcal{W}} c_\omega M(\omega)$$

which, moreover, satisfies

c4) if $M(\omega) \in \mathbf{C}_<(V), c_\omega \neq 0$, then there is no continuation of V at ω.

If we denote, for each j, $\mathbb{M}[j,r] := \{M(\tau) : \tau = X_1^{a_1} \cdots X_r^{a_r} \in \mathcal{W}, a_1 = \cdots = a_{j-1} = 0 \neq a_j\} \subset \mathbb{M}$, then each element $\ell \in \mathrm{Span}_K(\mathbb{M} \setminus \{\mathrm{Id}\})$ can be uniquely expressed as

$$\ell = \ell^{(1)} + \cdots + \ell^{(j)} + \cdots + \ell^{(r)},$$

where $\ell^{(j)} \in \mathrm{Span}_K(\mathbb{M}[j,r])$ for each j

Let $M(t) \in \mathbf{C}_{<}(V) \cap \mathbb{M}[\kappa, r]$ and let $\ell_{\iota}{}^{(\kappa)}$ be such that

$$\rho_{\kappa}(\mathbf{T}_{<}(\ell_{\iota}^{(\kappa)}) = M(t).$$

For $\kappa \leq j \leq r$ we can define a suitable set $J(j)$ of indices i, $1 \leq i \leq s$

The following conditions are equivalent:

1. The elementary continuation $C_{V,t}$ exists;
2. there are values $a_{ji} \in K$, such that, for each μ

$$\sigma_{\mu}\rho_{\kappa}(\ell_{\iota}^{(\kappa)}) + \sum_{j=1}^{r} \sum_{i \in J(j)} a_{ji}\sigma_{\mu}\rho_{j}(\ell_{i}^{(j)}) \in V.$$

Moreover, if the above conditions are satisfied,

$$C_{V,t} = \rho_{\kappa}(\ell_{\iota}^{(\kappa)}) + \sum_{j=1}^{r} \sum_{i \in J(j)} a_{ji}\rho_{j}(\ell_{i}^{(j)}).$$

$\mathsf{I} \subset \mathsf{m}$ which is given by means of any set of generators
$F := \{f_1, \ldots, f_t\} \subset \mathsf{m}$
If $\{\ell_1, \ell_2, \ldots, \ell_s\}$ denotes the ordered Macaulay basis wrt $<$ of I, which we aim to compute, and, for any $i < s$, we set

- $V_i := \{\ell_1, \ell_2, \ldots, \ell_i\}$
- $C_i := \{\tau \in \mathbf{C}_<(V_i) : \text{ there is an elementary continuation of } V_i \text{ at } \tau\}$,

we know that, for each i, exists $c_\tau \in K$ such that
$\ell_{i+1} = \sum_{\tau \in C_i} c_\tau C_{V_i,\tau}$.

$$\ell_{i+1} \in \mathfrak{M}(\mathsf{I}) \iff ev(\ell_{i+1}) = \sum_{\tau \in C_i} c_\tau ev(C_{V_i,\tau}) = 0$$

Next episode:
The Return of Macaulay
(?)