On the computation of algebraic relations of bivariate polynomials
(and application to the moment problem)

work in progress by

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General context

Given a $K[\mathbf{X}]$–module $\mathcal{M} \subset K[\mathbf{X}]^n$, $\mathbf{X} = (X_1, \ldots, X_r)$
and elements $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in K[\mathbf{X}]^n/\mathcal{M}$

Compute elements $p_1, \ldots, p_m \in K[\mathbf{X}]$ such that

$$p_1\boldsymbol{f}_1 + \cdots + p_m\boldsymbol{f}_m = 0 \text{ in } K[\mathbf{X}]^n/\mathcal{M}$$

Often we use the compact notation $P\boldsymbol{F} \in \mathcal{M}$ with

$P = 1 \times m$ vector of $p_1, \ldots, p_m$

$\boldsymbol{F} = m \times 1$ column vector of (row vectors) $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m$
$= m \times n$ polynomial matrix, that is $\boldsymbol{F} \in K[\mathbf{X}]^{m\times n}$

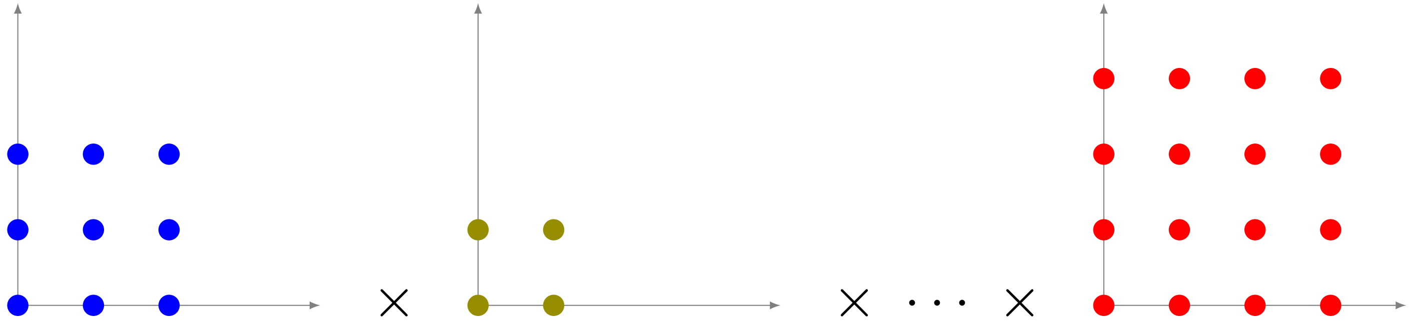
Concrete situation

$K[\mathbf{X}]^n/\mathcal{M}$ is a finite-dimensional $K-$vector space:

$$D = \dim_K(K[\mathbf{X}]^n/\mathcal{M}) < +\infty$$

For instance $\mathcal{M} = I_1 \times \cdots \times I_n$ with $I_i \subset K[\mathbf{X}]$ $0-$dim ideal

Interesting basic case: products of boxes



$$K[\mathbf{X}]^n/\mathcal{M} \sim \quad \times \quad \times \cdots \times$$

$$\mathcal{M} \sim \color{blue}{\langle X^3, Y^3 \rangle} \color{black}{\times} \color{olive}{\langle X^2, Y^2 \rangle} \color{black}{\times \cdots \times} \color{red}{\langle X^4, Y^4 \rangle}$$

Special cases

Rational reconstruction $[r=1, n=1]$

given $f, g \in K[X]_D$, find $p_1, p_2 \in K[X]_{D/2}$:

$$f = p_1/p_2 \bmod \langle g \rangle$$

(*Padé approximation* $g = X^D$)

Hermite-Padé approximation $[r=2, n=1]$

given $f_i \in K[X,Y]/\langle X^d, Y^d \rangle$, find $p_i \in K[X,Y]_{(d/\sqrt{m}, d/\sqrt{m})}$:

$$p_1 f_1 + \cdots + p_m f_m \in \langle X^d, Y^d \rangle$$

Modular inversion $[n=1]$

$$pf = 1 \text{ in } K[\mathbf{X}]/I$$

Structured linear algebra over K

Computing one relation $P \in K[\mathbf{X}]^m$ is a linear algebra problem : For instance, say we look for a, b satisfying

$$au + bv \in \left\langle X^D \right\rangle$$

given polynomials u, v. This corresponds to solving a truncated block-Hankel linear system over K

$$\begin{bmatrix} & & & u_0 & & & & v_0 \\ & & u_0 & u_1 & & & v_0 & v_1 \\ & \iddots & & \vdots & & \iddots & & \vdots \\ u_0 & & & & v_0 & & & \\ \vdots & & & \vdots & \vdots & & & \vdots \\ u_{D/2-1} & \cdots & u_{D-2} & u_{D-1} & v_{D/2-1} & \cdots & v_{D-2} & v_{D-1} \end{bmatrix} \begin{bmatrix} a_{D/2} \\ \vdots \\ a_0 \\ b_{D/2} \\ \vdots \\ b_0 \end{bmatrix} = 0$$

Structured linear algebra over K

$$au + \cdots \in \left\langle X^d, Y^d \right\rangle$$

$$\begin{bmatrix} & & & & & & & & u_{00} \\ & & & & & & & u_{00} & u_{10} \\ & & & & & & u_{00} & u_{10} & u_{20} \\ & & & & & u_{00} & & & u_{01} \\ & & & & u_{00} & u_{10} & & u_{01} & u_{11} \\ & & & u_{00} & u_{10} & u_{20} & u_{01} & u_{11} & u_{21} \\ & & u_{00} & & & u_{01} & & & u_{02} \\ & u_{00} & u_{10} & & u_{01} & u_{11} & & u_{02} & u_{12} \\ u_{00} & u_{10} & u_{20} & u_{01} & u_{11} & u_{21} & u_{02} & u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} a_{22} \\ a_{12} \\ a_{02} \\ a_{21} \\ a_{11} \\ a_{01} \\ a_{20} \\ a_{10} \\ a_{00} \end{bmatrix} = 0$$

Recently : Bostan-Jeannerod-Mouilleron-Schost '17

↪ $O^{\sim}(\alpha^{\omega-1} D)$ matrices with displacement rank α

Algorithmic problem

The kernel of the morphism of modules

$$\begin{array}{rrcl} \varphi : & K[\mathbf{X}]^m & \longrightarrow & K[\mathbf{X}]^n/\mathcal{M} \\ & (p_1, \ldots, p_m) & \longmapsto & p_1 \boldsymbol{f}_1 + \cdots + p_m \boldsymbol{f}_m \end{array}$$

is the module of relations mod $\mathcal{M}$:

$$\mathrm{Rel}_{\mathcal{M}}(\boldsymbol{F}) = \left\{ P = (p_1, \ldots, p_m) \in K[\mathbf{X}]^m : P\boldsymbol{F} \in \mathcal{M} \right\}$$

It holds that $\dim(K[\mathbf{X}]^m/\mathrm{Rel}_{\mathcal{M}}(\boldsymbol{F})) \leq D < +\infty$.

General goal :

compute a Groebner basis of $\mathrm{Rel}_{\mathcal{M}}(\boldsymbol{F}) = \ker \varphi$

for a given term order $\prec$ (that may be chosen according to degree constraints on $p_1, \ldots, p_m$)

Related work

$D = \dim_K(K[\mathbf{X}]^n/\mathcal{M})$
r = numb. of var. $X_1, \ldots, X_r$

Change of monomial ordering

FGLM '93	$O(r\,D^3)$	
FGHR '14, Neiger '16	$O^{\sim}(r\,D^{\omega})$	(assuming mult. mat.)

Dual description via functionals

BM '82, MMM '92	$O(r\,D^3 + f r D^2)$	(D functionals)
O'Keeffe-Fitzpatrick '97,'02	$O(r\,D^3)$	(appl. coding theory)

Univariate case

Beckermann-Labahn '94	$O^{\sim}(m^{\omega}\,D)$	(Hermite-Padé X^D)
Neiger-Vu '17	$O^{\sim}(m^{\omega-1}\,D)$	(in-out size $O(m\,D)$)

Example

Consider the Padé approximation over $K = \mathbb{Z}/5\mathbb{Z}$:

$$P\boldsymbol{F} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} 2X^4 - 4X - 1 \\ 3X^5 + X^4 + X^3 - X - 1 \\ X - 1 \\ 2X^2 + X + 1 \end{bmatrix} \bmod \langle X^6 \rangle$$

A trivial relation : $P = \begin{bmatrix} X^6 & 0 & 0 & 0 \end{bmatrix}$

A GB (TOP order) of $\mathrm{Rel}_{\langle X^6 \rangle}(F)$ is at most quadratic :

$$P = \begin{bmatrix} -X-1 & 2X & X+2 & 1 \\ 2X & 2X-2 & -1 & X+2 \\ X^2 - X - 2 & -1 & -1 & 1 \\ 2X+1 & X^2+2X+1 & 1 & -2 \end{bmatrix}$$

A structured example

We want to compute all relations between the rows of

$$F = \begin{bmatrix} 2 & a+b & a^2+b^2 & a^3+b^3 \\ a+b & a^2+b^2 & a^3+b^3 & a^4+b^4 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 & a^5+b^5 \\ a^3+b^3 & a^4+b^4 & a^5+b^5 & a^6+b^6 \end{bmatrix}$$

For $\mathcal{M} = K[a,b]^4$ a GB is given by

$$P = \begin{bmatrix} ab & -a-b & 1 & \\ & ab & -a-b & 1 \end{bmatrix} = \begin{bmatrix} ab\,\mathbf{e}_1 - (a+b)\mathbf{e}_2 + \mathbf{e}_3 \\ ab\,\mathbf{e}_2 - (a+b)\mathbf{e}_3 + \mathbf{e}_4 \end{bmatrix}$$

Morally, we compute only 1 element : $ab\,\mathbf{e}_i - (a+b)\mathbf{e}_{i+1} + \mathbf{e}_{i+2}$.

Towards a divide-and-conquer algorithm

Suppose:

1. $\mathcal{M}_2 \subset \mathcal{M}_1 \subset K[\mathbf{X}]^m$ are $K[\mathbf{X}]$−modules.

$\mathcal{M}_2 = \langle X^4, Y^2\rangle \subset \langle X^2, Y^2\rangle = \mathcal{M}_1$

2. $\mathscr{G}_1 = \{P_1, \ldots, P_{m_1}\}$: $\langle \mathscr{G}_1 \rangle = \mathrm{Rel}_{\mathcal{M}_1}(\boldsymbol{F})$

$P_1\boldsymbol{F} \in \langle X^2, Y^2\rangle, \ldots, P_{m_1}\boldsymbol{F} \in \langle X^2, Y^2\rangle$

3. $\mathscr{G}_2 = \{Q_1, \ldots, Q_{m_2}\}$ such that $\langle \mathscr{G}_2 \rangle = \mathrm{Rel}_{\mathcal{M}_2}(\mathscr{G}_1\boldsymbol{F})$

Then $\langle \mathscr{G}_2\mathscr{G}_1 \rangle = \mathrm{Rel}_{\mathcal{M}_2}(\boldsymbol{F})$. Indeed, for $R \in \mathrm{Rel}_{\mathcal{M}_2}(\boldsymbol{F})$:

$Q_iP_j\boldsymbol{F} \in \langle X^4, Y^2\rangle$

$$R\boldsymbol{F} \in \mathcal{M}_2 \subset \mathcal{M}_1 \Rightarrow R = \Lambda\mathscr{G}_1 = \xi\mathscr{G}_2\mathscr{G}_1.$$

An example

$$\boldsymbol{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} X^2 - 1 \\ Y - X^3 \end{bmatrix} \qquad \mathcal{M}_2 = \langle X^4, Y^2 \rangle \subset \langle X^2, Y^2 \rangle = \mathcal{M}_1$$

One has

$$\mathscr{G}_1 = \begin{bmatrix} Y & 1 \\ X^2 & 0 \\ 0 & Y \\ 0 & X^2 \end{bmatrix} \qquad \mathscr{G}_1 \boldsymbol{F} = \begin{bmatrix} YX^2 - X^3 \\ X^4 - X^2 \\ Y^2 - YX^3 \\ X^2Y - X^5 \end{bmatrix} \overset{\mathcal{M}_2}{=} \begin{bmatrix} YX^2 - X^3 \\ -X^2 \\ -YX^3 \\ X^2Y \end{bmatrix}$$

and

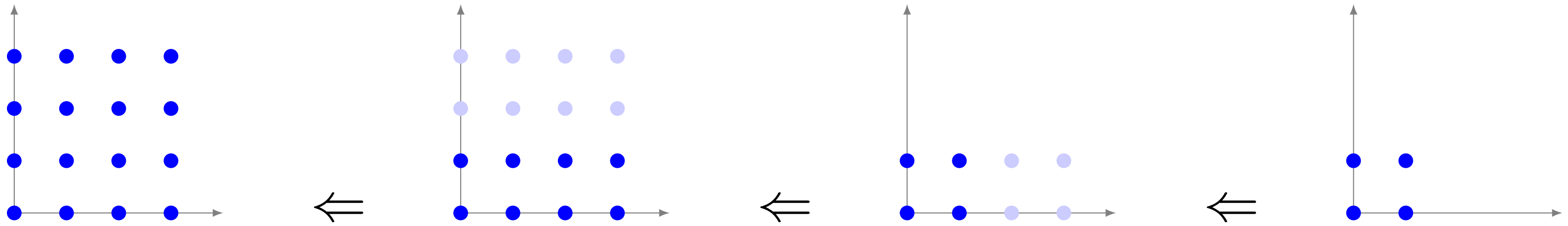
$$\mathscr{G}_2 = \begin{bmatrix} Y & 0 & -1 & 0 \\ X & 0 & 1 & 0 \\ 0 & Y & 0 & 1 \\ -1 & X & 0 & 1 \\ 0 & 0 & Y & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & Y \\ 0 & 0 & 1 & X \end{bmatrix} \qquad \mathscr{G}_2\mathscr{G}_1 = \begin{bmatrix} Y^2 & 0 \\ XY & X+Y \\ X^2Y & X^2 \\ X^3-Y & X^2-1 \\ 0 & Y^2 \\ 0 & XY \\ 0 & X^2Y \\ 0 & X^3+Y \end{bmatrix}$$

In this special case:

- $\mathscr{G}_1$ and $\mathscr{G}_2$ are TOP-GRLEX GB
- $\mathscr{G}_2\mathscr{G}_1$ is a (not reduced) TOP-GRLEX GB

We used a strategy that works for the case $r = 1$ (shifts)

Bivariate divide-and-conquer



Sketch of the algorithm (Groebner basis of $\mathrm{Rel}_{\langle X^d,Y^e\rangle}(\boldsymbol{F})$)

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GBRel($(d,e)$,$\boldsymbol{F}$,$\prec$)                                      # top algorithm
    If $d = e = 1$ then BaseCase($\boldsymbol{F}$,$\prec$)        # base case mod $\langle X, Y\rangle$
    Else if $d > e$ then $\mathscr{G}_1 \leftarrow$ GBRel($(d/2,e)$,$\boldsymbol{F}$,$\prec$)    # recursion
                         $\boldsymbol{G} \leftarrow X^{-d/2}\mathscr{G}_1\boldsymbol{F}$                   # residual
                         $\prec_2 \leftarrow$ TO($\prec$)                # update term order
                         $\mathscr{G}_2 \leftarrow$ GBRel($(d/2,e)$,$\boldsymbol{G}$,$\prec_2$)      # residual
                         Return $\mathscr{G}_2\mathscr{G}_1$.
        Else $\mathscr{G}_1 \leftarrow$ GBRel($(d,e/2)$,$\boldsymbol{F}$,$\prec$)               # recursion
             $\boldsymbol{G} \leftarrow Y^{-e/2}\mathscr{G}_1\boldsymbol{F}$                              # residual
             ...
Return $\mathscr{G}_2\mathscr{G}_1$.
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Moment problem

Given a multi-sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$, compute a set $S \subset \mathbb{R}^n$ and a measure μ with support in S and such that

$$y_\alpha = \int_S X_1^{\alpha_1} \cdots X_n^{\alpha_n} \, \mathrm{d}\mu \quad \forall \, \alpha \in \mathbb{N}^n$$

Inverse problem with origin in *functional analysis*, crucial applications in *polynomial optimization*, *control theory*...

Example : $\mu = \delta_a + \delta_b$ will give the sequence of moments

$$y = (2, a+b, a^2+b^2, a^3+b^3, \ldots)$$

that can be represented with the rank-2 polynomial matrix

$$\left(\int_S X^{i+j} \, \mathrm{d}\mu \right) = \begin{bmatrix} 2 & a+b & a^2+b^2 & a^3+b^3 \\ a+b & a^2+b^2 & a^3+b^3 & a^4+b^4 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 & a^5+b^5 \\ a^3+b^3 & a^4+b^4 & a^5+b^5 & a^6+b^6 \end{bmatrix}$$

Moment problem (continued)

In our context, we computed the GB of the rows of the matrix:

$$ab\,\mathbf{e}_1 - (a+b)\mathbf{e}_2 + \mathbf{e}_3$$

This element of the GB corresponds to the univariate polynomial that vanishes over the support of the solution measure :

$$T \mapsto ab - (a+b)T + T^2 = (T-a)(T-b)$$

Algebraic relation = solution to the **symbolic moment problem**

Open questions :

Size of the GB for the symbolic moment problem ?

Sufficient to solve it modulo a special $\mathcal{M} \subset K[a,b]^4$?

Complexity of the symbolic (modular) MP ?

Conclusion: What to retain?

Main problem : computing GB bases of relations mod $\mathcal{M}$

Contains as special case classical problems such as Hermite-Padé approximation, interpolation, modular inversion problems.

Some approaches that yield advances in the univariate case could be extended in many variables, for instance divide-and-conquer techniques in boxes $(K[\mathbf{X}]/\langle X^d, Y^d\rangle)^n$.

Open questions concern complexity bounds for the general bivariate or multi-variate problem (hoped $O\left((\#\mathscr{G})^{\omega-1}\, D\right)$), or structured situations such as the modular MP.

Special cases

Bivariate interpolation $[r = 2, n = 1]$

given $\{(x_1, y_1), \ldots, (x_D, y_D)\} \subset K^2$, $x_i \neq x_j$:

$$\text{find } Q(X,Y) \text{ s.t. } \quad Q(x_i, y_i) = 0, i = 1, \ldots, D$$

In the basis $\{\prod(X - x_i), Y - Lag(X)\}$ but the bi-degrees of the polynomials are unbalanced: $(D, 0)$ and $(D - 1, 1)$.

Can we get Q with $\deg Q \leq (\sqrt{D}, \sqrt{D})$?

This can be cast as a problem of type : find $Q_i(X)$ s.t.

$$Q_0 + Q_1(X)L(X) + \cdots + Q_\ell(X)L(X)^\ell \mod \langle \prod(X - x_i) \rangle .$$