Special Properties of Zero-Dimensional Ideals: new Algorithms

Lorenzo Robbiano

University of Genoa, Italy
Department of Mathematics

General Sources

A: Writing a good program
is like writing a good paper,
but a program also needs to be fast.
(John Abbott)

B: Writing a good program
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(Anna Maria Bigatti)

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Affine or Projective?

given the choice between two theories,
take the one which is funnier

There are at least three serious reasons why we choose the affine setting.

- into local rings.

- The structure of the coordinate ring and its canonical module can be described via multiplication matrices.

- The affine setup is suitable for generalizing everything to families of 0-dimensional ideals via the border basis scheme.

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The Setting, I

- We let K be a field and R a 0-dimensional affine K-algebra, i.e. $R = P/I$, where $P = K[x_1, \dots, x_n]$ is a polynomial ring over K and I is a 0-dimensional ideal in P, hence $\dim_K(R) < \infty$.

- The ideal I has a primary decomposition of the form $I = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_s$ where [illegible]

- The image of $\mathfrak{Q}_i$ in R is denoted by $\mathfrak{q}_i$, and for the image of $\mathfrak{M}_i$ in R we write $\mathfrak{m}_i$. Then we have $\langle 0 \rangle = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, and $\mathfrak{m}_i = \operatorname{Rad}(\mathfrak{q}_i)$ for $i = 1, \dots, s$.

- We have $\imath : R \cong R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_s$ which is called the decomposition of R into local rings. For $i = 1, \dots, s$, the ring $R_i = R/\mathfrak{q}_i$ is a 0-dimensional local K-algebra with maximal ideal $\bar{\mathfrak{m}}_i = \mathfrak{m}_i/\mathfrak{q}_i$. The ideal $\operatorname{Soc}(R_i) = \operatorname{Ann}_{R_i}(\bar{\mathfrak{m}}_i)$ is called the socle of R_i. For all fields $L_i = R_i/\bar{\mathfrak{m}}_i \cong R/\mathfrak{m}_i$ we put $\ell_i = \dim_K(L_i)$.

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The Setting, II

We equip P with the (standard) degree filtration $\widetilde{\mathcal{F}} = (F_i P)_{i \in \mathbb{Z}}$, where

$$F_i P \;=\; \{f \in P \mid \deg(f) \le i\} \;\cup\; \{0\}$$

For every $i \in \mathbb{Z}$, let $F_i I = F_i P \cap I$, and let $F_i R = F_i P / F_i I$. Then the family $(F_i I)_{i \in \mathbb{Z}}$ is called the induced filtration on I, and the family $\mathcal{F} = (F_i R)_{i \in \mathbb{Z}}$ is a $\mathbb{Z}$-filtration on R which is called the degree filtration on R.

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Let $R = P/I$ be a 0-dimensional affine K-algebra as above.

(a) The map $\mathrm{HF}^a_R : \mathbb{Z} \longrightarrow \mathbb{Z}$ where $i \mapsto \dim_K(F_iR)$ is called the affine Hilbert function of $\mathbb{X}$.

(b) The number $\mathrm{ri}(R) = \min\{i \in \mathbb{Z} \mid \mathrm{HF}^a_{\mathbb{X}}(j) = \dim_K(R) \text{ for all } j \geq i\}$ is called the regularity index of R.

(c) The first difference function $\Delta\,\mathrm{HF}^a_R(i) = \mathrm{HF}^a_R(i) - \mathrm{HF}^a_R(i-1)$ of HF^a_R is called the Castelnuovo function of R, and $\Delta_R = \Delta\,\mathrm{HF}^a_R(\mathrm{ri}(R))$ is the last difference of R.

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Example

Let $I = \langle x^3, y^3 \rangle$. Then we have $\Delta\,\mathrm{HF}_R^a = (1, 2, 3, 2, 1)$. Therefore we have $\mathrm{ri}(R) = 4$ and $\Delta_R = 1$.

The same for two generic (no common factor) cubics.

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A regular sequence of topics... in no particular order

Complete Intersections

or

Regular Sequences

H. Wiebe, *Über homologische Invarianten lokaler Ringe*, Math. Ann. 179 (1969), 257-274.

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H. Wiebe, *Über homologische Invarianten lokaler Ringe*, Math. Ann. 179 (1969), 257-274.

Complete Intersections

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A Result of Wiebe

Recall that a 0-dimensional local ring of the form $K[x_1, \ldots, x_n]_{\mathfrak{M}}/I$ with a field K, a maximal ideal $\mathfrak{M}$, and a 0-dimensional ideal I, is called a complete intersection if I can be generated by a regular sequence of length n.

The i-th Fitting ideal of a module M is denoted by $\mathrm{Fitt}_i(M)$.

Recall that a maximal ideal in the polynomial ring can always be generated by a regular sequence of length n (see [3], Cor. 5.3.14).

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Proposition (Wiebe)

A local ring R with maximal ideal $\mathfrak{m}$ is a 0-dimensional complete intersection if and only if the 0-th Fitting ideal of $\mathfrak{m}$ satisfies $\mathrm{Fitt}_0(\mathfrak{m}) \neq \langle 0 \rangle$.

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Proposition (Wiebe)

Let $P = K[x_1, \ldots, x_n]$, let $\mathfrak{M}$ be a maximal ideal of P, let $\{g_1, \ldots, g_n\}$ be a system of generators of $\mathfrak{M}$, let $I \subset P$ be an $\mathfrak{M}$-primary ideal, let $\{f_1, \ldots, f_r\}$ be a system of generators of I, let $R = P/I$, and let $\mathfrak{m} = \mathfrak{M}/I$.
For $i = 1, \ldots, r$, write $f_i = \sum_{j=1}^{n} a_{ij} g_j$, and form the matrix $\overline{W} \in \mathrm{Mat}_{n,r}(R)$ of size $n \times r$ whose columns are given by $\sum_{j=1}^{n} \bar{a}_{ij} e_j$ for $i = 1, \ldots, r$, where $\bar{a}_{ij}$ denotes the residue class of a_{ij} in R.

Then the 0-th Fitting ideal $\mathrm{Fitt}_0(\mathfrak{m})$ is generated by the minors of order n of $\overline{W}$.

An Algorithm

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Algorithm (Checking Local Complete Intersection Schemes)

Let $\mathbb{X}$ *be a 0-dimensional scheme in* $\mathbb{A}^n$*, let* $R_{\mathbb{X}} = P/I_{\mathbb{X}}$ *be the affine coordinate ring of* $\mathbb{X}$*, and let* $\mu = \dim_K(R_{\mathbb{X}})$*. The following instructions define an algorithm which checks whether* $\mathbb{X}$ *is a local complete intersection and returns the corresponding Boolean value.*

(1) *Compute the primary decomposition* $I_{\mathbb{X}} = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_s$ *of the ideal* $I_{\mathbb{X}}$*, where* $\mathfrak{Q}_i = \langle f_{i1}, \ldots, f_{i\nu_i} \rangle$ *is a primary ideal in* P *and* $f_{i1}, \ldots, f_{i\nu_i} \in P$ *for* $i = 1, \ldots, s$*.*

(2) *For* $i = 1, \ldots, s$*, check whether* $P/\mathfrak{Q}_i$ *is a local complete intersection ring using the following commands. If the answer is always* `TRUE(i)`*, return* `TRUE` *and stop.*

(3) *Compute a regular sequence* $(g_{i1}, \ldots, g_{in}) \in P^n$ *which generates the maximal ideal* $\mathfrak{M}_i = \mathrm{Rad}(\mathfrak{Q}_i)$*.*

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Let us use our Algorithm to check whether $\mathbb{X}$ is a local complete intersection scheme. The calculation of the primary decomposition of $I_{\mathbb{X}}$ yields that $I_{\mathbb{X}}$ is a primary ideal and its radical is the maximal ideal $\mathfrak{M} = \langle x - z,\ y - z^2,\ z^3 - z - 1 \rangle$. Here the polynomials $g_1 = x - z$, $g_2 = y - z^2$, and $g_3 = z^3 - z - 1$ form a regular sequence which generates $\mathfrak{M}$.

Thus we represent $f_1, \ldots, f_4$ as required by Step (4) of the algorithm and get the matrix

$$W = \begin{pmatrix} 0 & x - z & 0 & 0 \\ -1 & 1 & z & z^2 + y - 1 \\ 0 & 0 & 1 & z \end{pmatrix}$$

The tuple of residue classes in $P/I_{\mathbb{X}}$ of the minors of order 3 of W is $(\bar{x} - \bar{z},\ \bar{x}\bar{z} - \bar{y},\ 0,\ -\bar{x}\bar{y} + \bar{x} + 1)$. Therefore the scheme $\mathbb{X}$ is a local complete intersection.

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Gorenstein Rings

Figure: Daniel-Gorenstein (1923 –1992)

Why Gorenstein?

From an article of Craig Huneke (1997)

- Daniel Gorenstein is famous for his role in the classification of finite simple groups.
- A question occurring to everyone who studies Gorenstein rings is: Why are they called Gorenstein rings?
- His name being attached to this concept goes back to his thesis on plane curves, written under Oscar Zariski and published in the Transactions of the American
- They could perhaps more justifiably be called Bass rings, or Grothendieck rings, or Rosenlicht rings, or Serre rings. The usual definition now used in most textbooks goes back to the work of Bass in the paper
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but I don't know how to throw the old one away

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Definition and Remarks

The following remark provides a large class of locally Gorenstein rings.

Definition and Remarks

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Let R be a zero-dimensional affine K-algebra.

(a) Let $(R, \mathfrak{m})$ be a local ring. We say that R is a Gorenstein local ring if we have $\dim_{R/\mathfrak{m}}(\mathrm{Soc}(R)) = 1$.

(b) Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the primary components of the zero ideal in R. We say that R is a locally Gorenstein ring if $R/\mathfrak{q}_i$ is a locally Gorenstein local ring for $i = 1, \dots, s$.

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Remark

A field is clearly a Gorenstein ring. Consequently, every reduced zero-dimensional affine K-algebra R is a locally Gorenstein ring, as we can see by applying the isomorphism $R \cong R/\mathfrak{m}_1 \times \cdots \times R/\mathfrak{m}_s$ induced by the primary decomposition of R.

Characterization of Locally Gorenstein K-algebras

The next theorem characterizes locally Gorenstein rings and provides a link between this property of the ring R and the commendability of its multiplication family $\mathcal{F}$.

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Theorem (Characterization of Zero-Dimensional Locally Gorenstein Algebras)

Let R be a zero-dimensional affine K-algebra. The following conditions are equivalent.

(a) *The ring R is a Gorenstein ring.*

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An Algorithm

With the help of this theorem, we can write down an algorithm which checks whether R is a locally Gorenstein ring.

Let $\mathbb{X}$ be a 0-dimensional scheme in $\mathbb{A}^n$, let $R_{\mathbb{X}} = P/I_{\mathbb{X}}$ be the affine coordinate ring of $\mathbb{X}$ and let $\rho = \dim_K(R_{\mathbb{X}})$. The following instructions define an algorithm which checks whether $\mathbb{X}$ is a locally Gorenstein scheme and returns the corresponding Boolean value.

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(1) *Compute a tuple of polynomials whose residue classes $B = (b_1, \ldots, b_\mu)$ form a K-basis of $R_{\mathbb{X}}$.*

(2) *For $i = 1, \ldots, n$, compute the matrix $M_{b_i} \in \mathrm{Mat}_\mu(K)$ representing the multiplication by b_i on R in the basis B.*

(3) *Let $z_1, \ldots, z_\mu$ be new indeterminates, and let $C \in \mathrm{Mat}_\mu(K[z_1, \ldots, z_\mu])$ be the matrix whose columns are $M_{b_i}^{\mathrm{tr}} \cdot (z_1, \ldots, z_\mu)^{\mathrm{tr}}$ for $i = 1, \ldots, \mu$.*

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An Example

Let us check whether R is locally Gorenstein or not. Note that the given generating set is the reduced Gröbner basis of I with respect to `DegRevLex`.
So, a K-basis B of R is given by the residue classes of the elements in the tuple $(1,\ z,\ y,\ x,\ z^2,\ yz,\ xz,\ y^2,\ xy)$.

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Let $K = \mathbb{Q}$, let $P = K[x,y,z]$, and let $R = P/I$, where I is the ideal of P generated by

$$\begin{aligned}\{\, & x^2 - 18xy + 43y^2 + 12xz - \tfrac{170}{3}yz + \tfrac{218}{3}z^2 - 4x + \tfrac{340}{3}y - 216z + \tfrac{166}{3},\\ & xy^2 - 3xy - \tfrac{4}{9}y^2 + xz - \tfrac{32}{27}yz - \tfrac{28}{27}z^2 + \tfrac{64}{27}y + \tfrac{28}{9}z - \tfrac{32}{27},\\ & y^3 - \tfrac{17}{9}y^2 + \tfrac{17}{27}yz - \tfrac{2}{27}z^2 - \tfrac{88}{27}y + \tfrac{20}{9}z - \tfrac{10}{27},\\ & y^2z - \tfrac{10}{9}y^2 - \tfrac{17}{27}yz + \tfrac{83}{27}z^2 + \tfrac{34}{27}y - \tfrac{74}{9}z + \tfrac{64}{27},\\ & z^3 + \tfrac{2}{9}y^2 - \tfrac{11}{27}yz - \tfrac{40}{27}z^2 + \tfrac{22}{27}y - \tfrac{14}{9}z + \tfrac{16}{27},\\ & xz^2 - xy - \tfrac{1}{9}y^2 - \tfrac{8}{27}yz - \tfrac{7}{27}z^2 + \tfrac{16}{27}y + \tfrac{7}{9}z - \tfrac{8}{27},\\ & yz^2 + \tfrac{2}{9}y^2 - \tfrac{38}{27}yz - \tfrac{67}{27}z^2 - \tfrac{32}{27}y + \tfrac{49}{9}z - \tfrac{38}{27},\\ & xyz - \tfrac{1}{9}y^2 - 3xz - \tfrac{8}{27}yz - \tfrac{7}{27}z^2 + x + \tfrac{16}{27}y + \tfrac{7}{9}z - \tfrac{8}{27}\,\}.\end{aligned}$$

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As the computation of the determinant of the matrix $C \in K[z_1, \dots, z_9]$ of size 9×9 in Step (4) of the Algorithm is quite demanding, we substitute in C the indeterminates $(z_1, \dots, z_9)$ by the numbers $\lambda = (1, -3, -1, 2, 4, -1, -1, 1, 3)$ and get

$$C_\lambda = \begin{pmatrix} 1 & -3 & -1 & 2 & 4 & -1 & -1 & 1 & 3 \\ -3 & 4 & -1 & -1 & \frac{23}{27} & \frac{671}{27} & \frac{191}{27} & -\frac{1015}{27} & -\frac{25}{27} \\ -1 & -1 & 1 & 3 & \frac{671}{27} & -\frac{1015}{27} & -\frac{25}{27} & \frac{178}{27} & \frac{710}{27} \\ 2 & -1 & 3 & -\frac{2719}{3} & \frac{191}{27} & -\frac{25}{27} & \frac{108017}{27} & \frac{710}{27} & -\frac{107924}{27} \\ 4 & \frac{23}{27} & \frac{671}{27} & \frac{191}{27} & \frac{257}{9} & \frac{493}{27} & -\frac{25}{27} & \frac{338}{27} & \frac{200}{9} \\ -1 & \frac{671}{27} & -\frac{1015}{27} & -\frac{25}{27} & \frac{493}{27} & \frac{338}{27} & \frac{200}{9} & -\frac{2696}{27} & -\frac{266}{27} \\ -1 & \frac{191}{27} & -\frac{25}{27} & \frac{108017}{27} & -\frac{25}{27} & \frac{200}{9} & -\frac{35938}{9} & -\frac{266}{27} & \frac{348632}{27} \\ 1 & -\frac{1015}{27} & \frac{178}{27} & \frac{710}{27} & \frac{338}{27} & -\frac{2696}{27} & -\frac{266}{27} & \frac{1163}{27} & \frac{1715}{27} \\ 3 & -\frac{25}{27} & \frac{710}{27} & -\frac{107924}{27} & \frac{200}{9} & -\frac{266}{27} & \frac{348632}{27} & \frac{1715}{27} & -\frac{143783}{9} \end{pmatrix}$$

Since we have $\det(C_\lambda) = \frac{114824810760065082500447360}{10460353203} \neq 0$, we know that $\det(C) \neq 0$, and hence the ring R is locally Gorenstein.

It turns out that $I = \mathfrak{Q}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3$ where $\mathfrak{Q}_1 = \langle (x - y^3 - 1)^2,\ y - z^2,\ z^3 - 3z + 1 \rangle$, $\mathfrak{M}_2 = \langle x,\ y^2 - 2,\ z - 2 \rangle$, $\quad \mathfrak{M}_3 = \langle x - 1, y + 1, z \rangle$.

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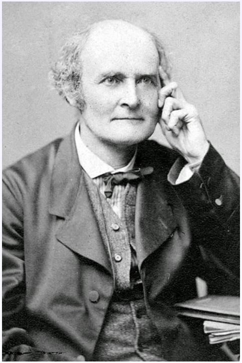
Cayley and Bacharach



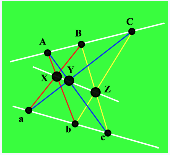
Arthur Cayley (1821 – 1895)

Isaak Bacharach (1854 – 1942)

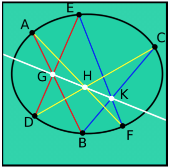
The Story Begins

even the longest journey
begins with the first step
(Chinese Proverb)

ca. 320: The theorem of Pappos[1] (Pappus Alexandrinus).



1640: The theorem of Pascal[2]



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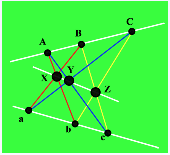

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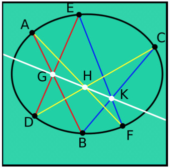

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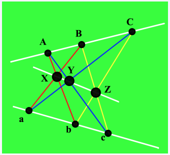

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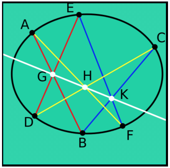

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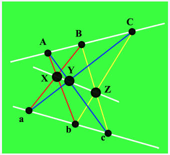

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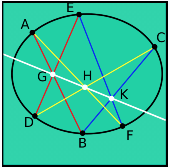

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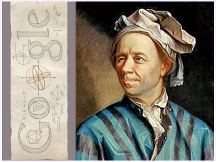
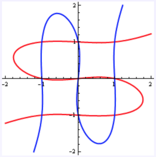
The Story Continues, I

1748: Cramer-Euler paradox.





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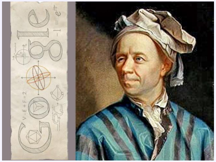

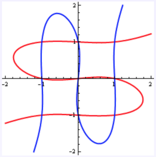

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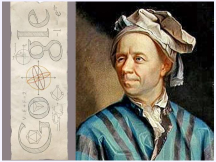

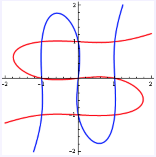

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The Modern Era, I

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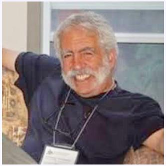

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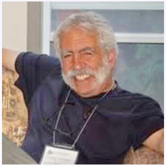 

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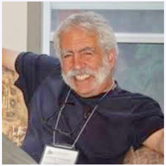

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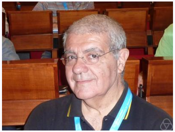

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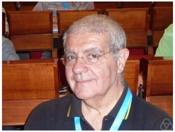

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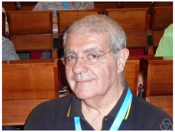
[15] G. Valla, Problems and Results on Hilbert Functions of Graded Algebras, Six lectures on Commutative Algebra, Elias, Giral, Miró-Roig, Zarzuela eds. Birkhäuser, 1998.

The Modern Era, III

1996:


D. Eisenbud, M. Green, J. Harris[13] reviewed the history of the CBP, put it in a general algebraic frame, and proposed conjectures and extensions (see also[14]).

1998:



G. Valla[15] discussed some of these conjectures and proved some partial results.

13 D. Eisenbud, M. Green, and J. Harris, Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. 33 (1996), 295-324.

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16 M. Kreuzer and L. Robbiano, *Computational Linear and Commutative Algebra*, Springer, Heidelberg, 2016.

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Let K be a field, let $P = K[x, y]$, and let $\mathfrak{Q} = \langle x^2, y^2 \rangle$. Clearly, the ideal $\mathfrak{Q}$ is $\mathfrak{M}$-primary for $\mathfrak{M} = \langle x, y \rangle$, and we have $\ell = \dim_K(P/\mathfrak{M}) = 1$.

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Now we consider the ideal $J_1 = \mathfrak{Q} + \langle x \rangle = \langle x, y^2 \rangle$. Clearly J_1 is $\mathfrak{M}$-primary and hence a $\mathfrak{Q}$-divisor of $\mathfrak{Q}$. Since we have $\dim_K(J_1/\mathfrak{Q}) = 2 > \ell$, the ideal J_1 is not a minimal $\mathfrak{Q}$-divisor of $\mathfrak{Q}$.

Next we look at the ideal $J_2 = \mathfrak{Q} + \langle xy \rangle = \langle x^2, xy, y^2 \rangle$. Again it is clear that J_2 is $\mathfrak{M}$-primary, and therefore a $\mathfrak{Q}$-divisor of $\mathfrak{Q}$. In this case we get the equality $\dim_K(J_2/\mathfrak{Q}) = 1 = \ell$, whence J_2 is even a minimal $\mathfrak{Q}$-divisor of $\mathfrak{Q}$.

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The following is a fundamental remark.

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For $f \in R \setminus \{0\}$, let $\mathrm{ord}_{\mathcal{F}}(f) = \min\{i \in \mathbb{Z} \mid f \in F_iR \setminus F_{i-1}R\}$. This number is called the order of f with respect to $\mathcal{F}$.

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Given a maximal ideal $\mathfrak{m}_i$ of R, the separators for $\mathfrak{m}_i$ may not be uniquely determined in two different ways:

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(2) *If $\dim_K(\mathrm{Soc}(R/\mathfrak{q}_i)) > \ell_i$, there exist separators f, g for $\mathfrak{m}_i$ which correspond to different $\mathfrak{Q}_i$-divisors of I. In this case, the ideals $\langle \bar{f} \rangle$ and $\langle \bar{g} \rangle$ in $R/\mathfrak{q}_i$ are not equal.*

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The Separator Degree

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Example

Let $K = \mathbb{Q}$, let $P = K[x, y]$, let $I = \langle xy,\ y^3,\ x^4 + x^2\rangle$, and let $R = P/I$.
We have $I = \mathfrak{Q}_1 \cap \mathfrak{Q}_2$, where $\mathfrak{Q}_1 = \langle y,\ x^2 + 1\rangle$ and $\mathfrak{Q}_2 = \langle xy,\ x^2,\ y^3\rangle$.
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The Cayley-Bacharach Property (CBP)

- There is an algorithm which checks whether the maximal ideal $\mathfrak{m}_i$ of R has maximal separator degree, i.e. $\operatorname{sepdeg}(\mathfrak{m}_i) = \operatorname{ri}(R)$.
- The ring R of the above example does not have the CBP.
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Proposition

The following conditions are equivalent.

(a) *For every minimal $\mathfrak{Q}_i$-divisor J of I and its image $\bar{J}$ in R, there is a generator f of $\bar{J}$ such that* $\mathrm{ord}_{\mathcal{F}}(f) = \mathrm{ri}(R)$.

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Let $R = P/I$ be a 0-dimensional affine K-algebra, and let $\mathbb{X} = \mathrm{Spec}(P/I)$ be the 0-dimensional affine scheme defined by I. We say that R has the Cayley-Bacharach property (CBP), or that $\mathbb{X}$ is a Cayley-Bacharach scheme, if the equivalent conditions of the above proposition are satisfied for all maximal ideals of R.

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The Canonical Module

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Definition

Let $R = P/I$ be a 0-dimensional affine K-algebra.

(a) If we equip the K-vector space $\omega_R = \mathrm{Hom}_K(R, K)$ with the R-module structure defined by $f \cdot \varphi(g) = \varphi(fg)$ for $f, g \in R$ and $\varphi \in \omega_R$, we obtain the canonical module of R.

(b) For every $i \in \mathbb{Z}$, let $G_i\omega_R = \{\varphi \in \omega_R \mid \varphi(F_{-i-1}R) = 0\}$. Then the family $\mathcal{G} = (G_i\omega_R)_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration of ω_R which we call the degree filtration of ω_R.

(c) The map $\mathrm{HF}^a_{\omega_R} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $\mathrm{HF}^a_{\omega_R}(i) = \dim_K(G_i\omega_R)$ for all $i \in \mathbb{Z}$ is called the affine Hilbert function of ω_R.

Remark

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Definition

Let $R = P/I$ be a 0-dimensional affine K-algebra.

(a) If we equip the K-vector space $\omega_R = \mathrm{Hom}_K(R, K)$ with the R-module structure defined by $f \cdot \varphi(g) = \varphi(fg)$ for $f, g \in R$ and $\varphi \in \omega_R$, we obtain the canonical module of R.

(b) For every $i \in \mathbb{Z}$, let $G_i\omega_R = \{\varphi \in \omega_R \mid \varphi(F_{-i-1}R) = 0\}$. Then the family $\mathcal{G} = (G_i\omega_R)_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration of ω_R which we call the degree filtration of ω_R.

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Let $R = P/I$ be a 0-dimensional affine K-algebra. TFAE

(a) *The ring R has the Cayley-Bacharach property.*

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Let $K = \mathbb{Q}$, let $P = K[x, y, z]$, let I be the ideal of P generated by $\{z^2 - x + 2z,\ xz - 2x - y + 4z,\ y^2 - x + z,\ x^2 - yz - 4x - 4y + 8z\}$, and let $R = P/I$.
The primary decomposition of I is $I = \mathfrak{M}_1 \cap \mathfrak{M}_2$, where we have $\mathfrak{M}_1 = \langle x,\ y,\ z\rangle$ and $\mathfrak{M}_2 = \langle z^2-x+2z,\ xz-2x-y+4z,\ y^2-x+z,\ x^2-yz-4x-4y+8z,\ xy-2yz-z-1\rangle$. Here $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are maximal ideals, $\mathfrak{M}_1$ is a linear maximal ideal, and $\mathfrak{M}_2$ corresponds to a residue field extension $K \subset L_2$ of degree 5. The affine Hilbert function of R is $(1, 4, 6, 6, \ldots)$, and hence $\mathrm{ri}(R) = 2$. A degree filtered K-basis of R is given by the residue classes of $\{1,\ z,\ y,\ x,\ yz,\ xy\}$. Thus we have $d = 6$ and $\Delta_R = 2$. The two matrices V_1 and V_2 computed in Step (3) of the algorithm are

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 & -8 \\ -1 & -2 & 0 & -4 & 1 & 1 \\ 0 & -4 & 1 & -8 & 1 & -2 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 4 & 1 & 5 \end{pmatrix}$$

Since the matrix $W = \mathrm{Col}(V_1, V_2)$ has a trivial kernel, we conclude that R has the CBP.

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Let $K = \mathbb{Q}$, let $P = K[x, y, z]$, let I be the ideal of P generated by $\{z^2 - x + 2z,\ xz - 2x - y + 4z,\ y^2 - x + z,\ x^2 - yz - 4x - 4y + 8z\}$, and let $R = P/I$. The primary decomposition of I is $I = \mathfrak{M}_1 \cap \mathfrak{M}_2$, where we have $\mathfrak{M}_1 = \langle x,\ y,\ z \rangle$ and $\mathfrak{M}_2 = \langle z^2 - x + 2z,\ xz - 2x - y + 4z,\ y^2 - x + z,\ x^2 - yz - 4x - 4y + 8z,\ xy - 2yz - z - 1 \rangle$. Here $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are maximal ideals, $\mathfrak{M}_1$ is a linear maximal ideal, and $\mathfrak{M}_2$ corresponds to a residue field extension $K \subset L_2$ of degree 5. The affine Hilbert function of R is $(1, 4, 6, 6, \dots)$, and hence $\mathrm{ri}(R) = 2$. A degree filtered K-basis of R is given by the residue classes of $\{1,\ z,\ y,\ x,\ yz,\ xy\}$. Thus we have $d = 6$ and $\Delta_R = 2$. The two matrices V_1 and V_2 computed in Step (3) of the algorithm are

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 & -8 \\ -1 & -2 & 0 & -4 & 1 & 1 \\ 0 & -4 & 1 & -8 & 1 & -2 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 4 & 1 & 5 \end{pmatrix}$$

Since the matrix $W = \mathrm{Col}(V_1, V_2)$ has a trivial kernel, we conclude that R has the CBP.

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Answer: *Pi*

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