Reading course report, on the book
Ideals, Varieties and Algorithms
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Abstract

This report gives an overview of the main ideas in Chapter 9 of the studied book, about the dimension of a variety. After recalling some definitions and basic properties about projective varieties, homogeneous ideals and Gröbner bases, we will begin our study of the dimension of a variety with the observation of a geometric way to define the dimension of a variety defined by a monomial ideal. Then, we show how to characterize this dimension by algebraic properties of the defining ideal. Using Gröbner bases to link the ideal of a variety to the monomial ideal of its leading terms, this allows us to give a satisfying definition to the dimension of any projective variety; we conclude with some properties linked to the dimension of a variety.

1 Ideals, varieties and Gröbner bases

In this report we will deal with *projective varieties*, that is, varieties in a projective space that are defined by *homogeneous ideals* of the ring of polynomials $k[x_0, \ldots, x_n]$, where k is an infinite field and $n \geqslant 1$. In this section, we briefly recall how one can define these two central objects — ideals and varieties — and how they are linked; we also give basic properties of Gröbner bases, which allow to effectively compute objects and effectively solve problems related to ideals and varieties.

1.1 Homogeneous ideals and projective varieties

Two points $(a_0, \ldots, a_n)$ and $(a'_0, \ldots, a'_n)$ in the affine space k^{n+1} are said to be equivalent if $(a_0, \ldots, a_n) = (\lambda a'_0, \ldots, \lambda a'_n)$ for some nonzero $\lambda \in k$; this defines an equivalence relation $\sim$ over $k^{n+1} - \{0\}$. In the projective space $\mathbb{P}^n(k)$, we work modulo $\sim$, which means that two equivalent points will be considered equal.

Definition 1. The *n-th dimensional projective space over k* is the set of equivalence classes $\mathbb{P}^n(k) = (k^{n+1} - \{0\})/\sim$. Thus each nonzero $(a_0, \ldots, a_n) \in k^{n+1}$ defines a point $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n(k)$.

A variety is basically defined as the set of common zeroes of some polynomials in $k[x_0, \ldots, x_n]$. For projective varieties, we restrict to *homogeneous* polynomials, for which vanishing at a point $(a_0, \ldots, a_n) \in k^{n+1} - \{0\}$ is equivalent to vanishing at any point $(\lambda a_0, \ldots, \lambda a_n)$ with $\lambda \neq 0$, which means that vanishing at a projective point $[a_0 : \cdots : a_n]$ is well-defined.

Definition 2. The *(projective) variety* defined by the homogeneous polynomials $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ is $\mathbf{V}(f_1, \ldots, f_m) = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(k) \mid f_i(a_0, \ldots, a_n) = 0 \text{ for } 1 \leqslant i \leqslant m\}$.

For example, linear subspaces of $\mathbb{P}^n(k)$ are projective varieties defined by a collection of polynomials that are homogeneous of degree 1. A variety $\mathbf{V}(f)$ defined by a single homogeneous polynomial f is called a hypersurface. Different sets of polynomials can yield the same variety: for instance, it is easily verified that $\mathbf{V}(x_0^2 + x_0x_1, x_1) = \mathbf{V}(x_0^2, x_1)$ because x_0^2 is a combination with polynomial coefficients of $x_0^2 + x_0x_1$ and x_1. More generally, any common zero of homogeneous polynomials $f_1, \ldots, f_m$ will also be a zero of any polynomial in the ideal $I = \langle f_1, \ldots, f_m \rangle$ generated by $f_1, \ldots, f_m$. This ideal has two remarkable features: it has a *finite* generating set consisting of *homogeneous* polynomials. The first feature is actually a general property: Hilbert's Basis Theorem says that every ideal of $k[x_0, \ldots, x_n]$ is finitely generated. Second, an ideal which has a generating set consisting of homogeneous polynomials is said to be a *homogeneous ideal* of $k[x_0, \ldots, x_n]$. Note that not all polynomials of a homogeneous ideal $I = \langle f_1, \ldots, f_m \rangle$ have to be homogeneous (e.g. $\langle x_0^2 + x_1, x_1 \rangle = \langle x_0^2, x_1 \rangle$), but that the homogeneous components of a polynomial $f \in I$ are themselves in I, so that it makes sense to say that a common zero of $f_1, \ldots, f_m$ is also a zero of any $f \in I$ even though f may not be homogeneous. This leads us to extend the definition of a variety: for I a homogeneous ideal, $\mathbf{V}(I) = \{p \in \mathbb{P}^n(k) \mid f(p) = 0 \text{ for all } f \in I\}$.

A related object if the set of polynomials which vanish on a variety.

Definition 3. Let $V \subset \mathbb{P}^n(k)$ be a (projective) variety. The *ideal of* V is defined as being the set $\mathbf{I}(V) = \{f \in k[x_0, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in V\}$. $\mathbf{I}(V)$ is a homogeneous ideal of $k[x_0, \ldots, x_n]$.

Let us now summarize the correspondence between homogeneous ideals and varieties.

Proposition 4. *$\mathbf{V}$: homogeneous ideals $\to$ varieties and $\mathbf{I}$: varieties $\to$ homogeneous ideals are inclusion-reversing, and for any variety $V \subset \mathbb{P}^n(k)$ we have $\mathbf{V}(\mathbf{I}(V)) = V$.*

In particular, $\mathbf{I}$ is one-to-one: two distinct varieties $V \neq W$ define two distinct homogeneous ideals $\mathbf{I}(V) \neq \mathbf{I}(W)$. One could hope for a reciprocal statement, but there are two clear hindrances. First, unwanted things happen on non-algebraically closed fields. Consider for instance the homogeneous ideals $\langle x_0, x_1 \rangle \neq \langle x_0^2 + x_1^2 \rangle$: over $k = \mathbb{R}$ the varieties $\mathbf{V}(x_0, x_1)$ and $\mathbf{V}(x_0^2 + x_1^2)$ will be equal, while over any algebraically closed field the two varieties are distinct. Second, varieties are inherently independent of any consideration of multiplicities of zeroes: for instance, whatever the ambient field k, the varieties $\mathbf{V}((x_0 - x_1)x_0)$ and $\mathbf{V}((x_0 - x_1)^4 x_0^3)$ will always be the same.

Given an ideal I of $k[x_0, \ldots, x_n]$ we define the *radical of* I as being the ideal $\sqrt{I} = \{f \in k[x_0, \ldots, x_n] \mid f^s \in I \text{ for some } s \geqslant 1\}$. I is said to be a *radical ideal* if $I = \sqrt{I}$. If I is homogeneous, then $\sqrt{I}$ is homogeneous and $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$, which somehow formalizes the statement above about ignoring multiplicities. We then have the following central result due to Hilbert.

Theorem 5 (Projective Nullstellensatz)**.** *Let k be an algebraically closed field and let I be a homogeneous ideal of $k[x_0, \ldots, x_n]$. If $V = \mathbf{V}(I)$ is a nonempty variety in $\mathbb{P}^n(k)$, then $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.*

Furthermore, $\mathbf{V}(I)$ is the empty variety if and only if $\langle x_0, \ldots, x_n \rangle \subset \sqrt{I}$, that is, for each $0 \leqslant i \leqslant n$, there is an integer $m_i \geqslant 0$ such that $x_i^{m_i} \in I$.

Obviously, $\langle x_0, \ldots, x_n \rangle \subset \sqrt{I}$ happens only when $I = \langle x_0, \ldots, x_n \rangle$ or $I = k[x_0, \ldots, x_n]$; otherwise, $I \subsetneq \langle x_0, \ldots, x_n \rangle$. Thus, over an algebraically closed field, the maps $\mathbf{I}$ and $\mathbf{V}$ are two inclusion-reversing bijections between the set of radical homogeneous ideals properly contained in $\langle x_0, \ldots, x_n \rangle$ and the set of nonempty projective varieties, which are inverses of each other.

1.2 Gröbner bases and algorithms

Let us now review basic facts about Gröbner bases, which allow us to solve some computational problems related to ideals or varieties.

Unlike in the ring of univariate polynomials $k[x]$, the ideals of $k[x_0, \ldots, x_n]$ (for $n \geqslant 1$) are not necessarily generated by a single polynomial, and as a consequence there is no proper generalization of the Euclidean division to multivariate polynomials. In the univariate case, algorithms to perform division with remainder or GCD computations. Among others, they are widely used to compute solutions to problems about ideals and varieties, such as testing ideal membership (via divisibility tests), computing ideal sum (via GCD computations), computing ideal intersection (via LCM computations), computing common roots of some polynomials (via GCD computations), computing a canonical representative of $f \in k[x]$ when working in some quotient ring $k[x]/\langle g \rangle$ (via division with remainder), etc. Some decades ago, the notion of *Gröbner basis* of an ideal of $k[x_0, \ldots, x_n]$ was introduced; it allows to perform a division algorithm in $k[x_0, \ldots, x_n]$ which has good properties, and thus helps to solve (in the multivariate case) algorithmic problems such as those mentioned above.

Here, we assume some familarity with the notion of *monomial order* of monomials in $k[x_0, \ldots, x_n]$, as well as for $f \in k[x_0, \ldots, x_n]$, the notions of *degree* of f, *total degree* of f, *leading monomial* $\mathrm{LM}(f)$, and *leading term* $\mathrm{LT}(f)$.

Fix a monomial order and some ideal $I = \langle f_1, \ldots, f_m \rangle$ of $k[x_0, \ldots, x_n]$. We have the following division with remainder in $k[x_0, \ldots, x_n]$. Given a polynomial $f \in k[x_0, \ldots, x_n]$, one can compute polynomials $a_1, \ldots, a_m, r$ such that $f = a_1 f_1 + \cdots + a_m f_m + r$, where $r = 0$ or none of the monomials appearing in r is divisible by one of $\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_m)$; in other words, r does not belong to the monomial ideal $\langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_m) \rangle$. However, without further assumption on the f_i's, uniqueness of the quotients and the remainder is not guaranteed, so that for instance we may have $r \neq 0$ and still $f \in I$, and the algorithm cannot be used for the central problem of testing ideal membership. Here come Gröbner bases into play; we denote by $\mathrm{LT}(I)$ the set of leading terms of polynomials in I, and $\langle \mathrm{LT}(I) \rangle$ the ideal they generate.

Definition 6. Having fixed a monomial order, a finite subset $\mathcal{G} = \{g_1, \ldots, g_m\}$ of an ideal I is said to be a Gröbner basis of I if $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m) \rangle$.

Every ideal $I \neq \{0\}$ has a Gröbner basis $\{g_1, \ldots, g_m\}$, which is also a basis of I, $I = \langle g_1, \ldots, g_m \rangle$. Now, given $f \in I$ we can compute as above the division $f = g + r$ with $g \in I$ and $r \in k[x_0, \ldots, x_n]$. Since $g_1, \ldots, g_m$ form a Gröbner basis of I we have that $r \notin \langle \mathrm{LT}(I) \rangle$, which ensures the uniqueness of the remainder r. Besides, an algorithm due to Buchberger allows to compute a Gröbner basis of an ideal I given by its generators $f_1, \ldots, f_m$. Now we are able to test ideal membership! This algorithm has been implemented in many computer algebra systems, the output being usually a *reduced* Gröbner basis of I, which is unique. Therefore, this gives a method to test whether two sets of polynomials generate the same ideal. Furthermore, there is also an algorithm which, on input I given by its generators, return a set of generators for $\sqrt{I}$. Thus, combining those two algorithms, over an algebraically closed field we have a method to test whether two sets of polynomials yield the same variety.

In the case of a linear system, a (reduced) Gröbner basis for a lexicographic order corresponds to a (reduced) row echelon form of the matrix of the system. In particular, a row echelon form allows to compute the dimension of the linear variety formed by solutions to the system; dimension is a very important invariant of a projective linear subspace. In what follows, we will extend this notion of dimension to projective varieties, give some properties and show how to compute it in some cases.

2 Dimension of varieties defined by a monomial ideal

We begin our study of the dimension of varieties with the special case of varieties defined by a monomial ideal. First, we will give a definition of dimension based on geometric properties of those varieties, and then we will see how dimension is linked to the so-called Hilbert polynomial of the ideal defining the variety. In what follows, the base field k is always infinite.

2.1 Geometric definition of the dimension

Here we are interested in varieties $\mathbf{V}(I)$ defined by a monomial ideal I, that is, an ideal for which there is a finite generating set consisting of monomials of $k[x_0, \dots, x_n]$. So we can write $I = \langle x^{\alpha_1}, \dots, x^{\alpha_m} \rangle$, where each α_i is a multi-index with nonnegative integer entries, and for any such index $\beta = (\beta_0, \dots, \beta_n)$ we write $x^\beta = x_0^{\beta_0} \cdots x_n^{\beta_n}$. A monomial ideal I is in particular a homogeneous ideal, because monomials are homogeneous polynomials.

For instance, let $I_0 = \langle x^2y, xz^3 \rangle$ in $k[x, y, z]$. Then, the corresponding (projective) variety $\mathbf{V}(I_0)$ in $\mathbb{P}_2(k)$ is the union of the two varieties $V_1 = \mathbf{V}(x)$ and $V_2 = \mathbf{V}(y, z)$. Indeed, using the identities $\mathbf{V}(fg) = \mathbf{V}(f) \cup \mathbf{V}(g)$ and $\mathbf{V}(f, g) = \mathbf{V}(f) \cap \mathbf{V}(g)$ for homogeneous polynomials f and g, we get $\mathbf{V}(I_0) = \mathbf{V}(x^2y) \cap \mathbf{V}(xz^3) = (\mathbf{V}(x) \cup \mathbf{V}(y)) \cap (\mathbf{V}(x) \cup \mathbf{V}(z)) = \mathbf{V}(x) \cup \mathbf{V}(y, z)$. The variety $V_1 = \{[0 : 1 : z], z \in k\} \cup \{[0 : 0 : 1]\}$ looks like a copy of $\mathbb{P}_1(k)$ inside $\mathbb{P}_2(k)$: this is a projective linear subspace of $\mathbb{P}_2(k)$ of dimension 1. It is not any linear subspace, but one which is defined by setting some coordinates to zero: we will call such a set a *projective coordinate subspace* of $\mathbb{P}_2(k)$. Similarly, $V_2 = \{[1 : 0 : 0]\}$ is a projective coordinate subspace of $\mathbb{P}_2(k)$ of dimension 0. It is easy to check that this kind of decomposition can be adapted to the variety of any monomial ideal, giving the following result.

Proposition 7. *The variety of a monomial ideal in $k[x_0, \dots, x_n]$ is a finite union of projective coordinate subspaces of $\mathbb{P}^n(k)$.*

This leads us to make the following definition for this type of varieties.

Definition 8. Let V be a variety which is a finite union of projective linear subspaces of $\mathbb{P}^n(k)$. Then the *dimension* of V, denoted by $\dim V$, is the largest of the dimensions of these subspaces.

For instance, the dimension of $\mathbf{V}(x^2y, xz^3)$ is 1. Let us check that this definition is sound. For a monomial ideal I in $k[x_0, \dots, x_n]$, suppose we have two decompositions of the variety $\mathbf{V}(I)$ as a finite union of projective linear subspaces $\mathbf{V}(I) = V_1 \cup \dots \cup V_s = W_1 \cup \dots \cup W_t$. We want to verify that $\max \dim V_i = \max \dim W_j$, so that $\dim \mathbf{V}(I)$ is well-defined. Indeed, if V_i has the largest dimension among the V_i's we have $V_i = (V_i \cap W_1) \cup \dots \cup (V_i \cap W_t)$, but since k is infinite, a union of linear subspaces of V_i cannot be the whole space V_i unless one of these subspaces is itself the whole space. In other words, $V_i = V_i \cap W_j$ for some j and thus $\dim W_j \geqslant \dim V_i$. So $\max \dim W_j \geqslant \max \dim V_i$, and by symmetry this concludes the verification.

Following the remarks before Proposition 7, here is an algorithm for computing the dimension of a variety defined by a monomial ideal $I = \langle x^{\alpha_1}, \dots, x^{\alpha_s} \rangle$. The idea of the algorithm is to compute the smallest number t of variables $x_{i_1}, \dots, x_{i_t}$ such that each monomial x^{α_i} in the generator set of I is divisible by at least one of those variables x_{i_j}. This implies that the projective coordinate subspace $\mathbf{V}(x_{i_1}, \dots, x_{i_t})$ of dimension $n-t$ is included in $\mathbf{V}(I)$, and therefore $\dim \mathbf{V}(I) \geqslant n-t$. For instance if the variable x_0 appears in every monomial x^{α_i}, it is clear that the subspace $\mathbf{V}(x_0)$ of dimension $n-1$ is included in $\mathbf{V}(I)$. One may carefully check that, because of the definition of dimension given above, this forces either $\dim \mathbf{V}(I) = n-1$ or $\mathbf{V}(I) = \mathbb{P}^n(k)$.

Going back to the general case, let us now explain why $n-t$ is actually equal to the dimension of $\mathbf{V}(I)$. Denoting $\dim \mathbf{V}(I) = n - t'$, there is some subspace of the form $\mathbf{V}(x_{j_1}, \dots, x_{j_{t'}})$ included in $\mathbf{V}(I)$. If some variable $x_{j_{i'}}$ does not divide some monomial x^{α_i} in the generator set defining I, then the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ with 1 at the $j_{i'}$ coordinate would be a zero of x^{α_i} but not of $x_{j_{i'}}$, which is a contradiction; thus each x^{α_i} is divisible by at least one of the $x_{j_0}, \dots, x_{j_{t'}}$. Now, since we have chosen the smallest such t, this forces $n - t \geqslant n - t' = \dim \mathbf{V}(I)$, and finally $n - t = \dim \mathbf{V}(I)$. Hence the correctness of the following algorithm.

Algorithm 1. DIMENSION OF A VARIETY DEFINED BY A MONOMIAL IDEAL

Input: a base field k, a positive integer n, monomials $x^{\alpha_1}, \dots, x^{\alpha_s}$ in $k[x_0, \dots, x_n]$

Output: the dimension of the variety $\mathbf{V}(x^{\alpha_1}, \dots, x^{\alpha_s})$ of $\mathbb{P}^n(k)$

1. For each $i \in \{1, \dots, s\}$ compute

$$M_i := \{j \in \{0, \dots, n\} \mid x_j \text{ divides } x^{\alpha_i}\}.$$

2. Compute

$$t := \min \{\mathrm{Card}(M),\ M \subset \{0, \dots, n\} \mid M \cap M_i \neq 0 \text{ for } 1 \leqslant i \leqslant s\}.$$

3. Return $n - t$ if it is nonnegative, report empty variety otherwise.

Note that t is well-defined and $t \leqslant n+1$, since the set $M = \{0, \dots, n\}$ always satisfies $M \cap M_i \neq 0$ for $1 \leqslant i \leqslant s$. Something special happens when $t = n+1$, since the set of generators $\{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ has to contain every variable to some positive exponent, $x_i^{\beta_i}$ with $\beta_i > 0$ for $0 \leqslant i \leqslant n$. (Indeed, if no monomial in the set of generators is a power of, say, x_0 then it is easy to see that $M = \{1, \dots, n\}$ satisfies $M \cap M_i \neq 0$ for $1 \leqslant i \leqslant s$, and thus we would have $t \leqslant n$.) But then, $\mathbf{V}(x^{\alpha_1}, \dots, x^{\alpha_s})$ is included in $\mathbf{V}(x_0^{\beta_0}, \dots, x_n^{\beta_n})$, which is the empty variety.

2.2 Exponents of monomials not in the ideal

Still focusing on monomial ideals, in this section we aim at showing links between some properties of the complement of the ideal and the variety of the ideal. More precisely, Proposition 9 will link some sets of exponents of monomials not in the ideal with projective coordinate subspaces that are included in the variety, giving a characterization of the dimension using the structure of the set of monomials not in the ideal.

Let us begin with an example. Let I be a monomial ideal in $k[x, y]$. It is convenient to represent the monomials of $k[x, y]$ by their exponents in $\mathbb{Z}^2_{\geqslant 0}$, and this will help us seeing the structure of

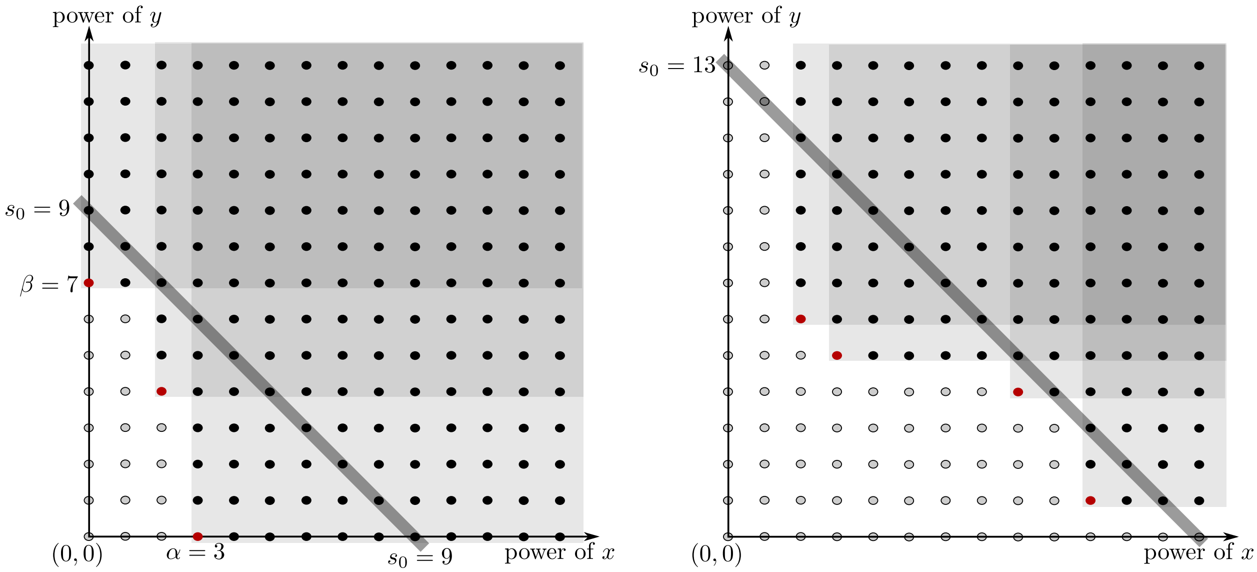

Figure 1: *Every dot represent the exponent in $\mathbb{Z}^2_{\geqslant 0}$ of a monomial in $k[x, y]$. Black dots are the exponents of the monomials in the ideal; grey dots are the exponents in $\mathcal{C}(I)$; red dots are the exponents of the monomials in the chosen generating set of the ideal.* Left: *$I = \langle x^3, x^2y^4, y^7 \rangle$, we are in situation (i). $\mathcal{C}(I)$ is finite,* $\mathrm{HF}_I(s) = 0$ *for $s \geqslant s_0$,* $\mathrm{HP}_I(s) = 0$ *and* $\mathbf{V}(I) = \emptyset \subset \mathbb{P}^1(k)$. Right: *$I = \langle x^2y^6, x^3y^5, x^8y^4, x^{10}y \rangle$, we are in situation (iv). $\mathcal{C}(I) = (0 \times \mathbb{Z}_{\geqslant 0}) \cup (1 \times \mathbb{Z}_{\geqslant 0}) \cup (\mathbb{Z}_{\geqslant 0} \times 0) \cup \{\text{finitely many points}\}$,* $\mathrm{HF}_I(s) = 3$ *for $s \geqslant s_0$,* $\mathrm{HP}_I(s) = 3$ *and* $\mathbf{V}(I) = \{[0:1], [1:0]\} \subset \mathbb{P}^1(k)$ *consists of finitely many points.*

the set of monomials not in I. We denote by $\mathcal{C}(I)$ the set of exponents in $\mathbb{Z}^2_{\geqslant 0}$ which correspond to monomials not in I. Four situations can arise (see Figure 1 for examples):

(i) $x^\alpha, y^\beta \in I$ for some positive exponents α, β. Then it is clear that $\mathcal{C}(I)$ is finite, its cardinality being upper bounded by $\alpha \cdot \beta$. Note that in this case $\mathbf{V}(I) = \emptyset$.

(ii) $x^\alpha \in I$ for some positive exponent α but no power of y is in I. Then $\mathcal{C}(I)$ surely contains the *line* $\{0\} \times \mathbb{Z}_{\geqslant 0}$, that is, all exponents of the form $(0, \beta)$ for $\beta \in \mathbb{Z}_{\geqslant 0}$. It may also contain some translates of this line, $\{i\} \times \mathbb{Z}_{\geqslant 0}$ for some $i \in \mathbb{Z}_{\geqslant 0}$; only a finite number of i's are possible since having $i \geqslant \alpha$ implies $x^i y^\beta \in I$. Finally, $\mathcal{C}(I)$ may also contain an additional finite number of points that are not on the line nor on the translates. In this case, $\mathbf{V}(I) = \{[0:1]\}$ is of dimension 0.

(iii) $y^\beta \in I$ for some positive exponent β but no power of x is in I. Like in the previous case, $\mathcal{C}(I)$ contains the line $\mathbb{Z}_{\geqslant 0} \times \{0\}$, and possibly a finite number of translates $\mathbb{Z}_{\geqslant 0} \times \{j\}$ and a finite number of points. $\mathbf{V}(I) = \{[1:0]\}$ is of dimension 0.

(iv) $x^\alpha y^\beta \in I$ for some positive exponents α, β, but no power of x or y is in I. Then $\mathcal{C}(I)$ contains the two lines $\mathbb{Z}_{\geqslant 0} \times \{0\}$ and $\{0\} \times \mathbb{Z}_{\geqslant 0}$ as well as possibly some translates $\mathbb{Z}_{\geqslant 0} \times \{i\}$ and $\{j\} \times \mathbb{Z}_{\geqslant 0}$ of these lines for a finite number of i's and j's (when $i \geqslant \alpha$ and $j \geqslant \beta$, we have $x^i y^j \in I$) and a finite number of points. In this case, $\mathbf{V}(I) = \{[0:1], [1:0]\}$ is again of dimension 0.

Thus, we see that $\mathcal{C}(I)$ and $\mathbf{V}(I)$ seem linked in some way. We will not list all possible cases for a monomial ideal I in $k[x, y, z]$, but let us look at a specific example which gives some more insight.

Consider the ideal $I_0 = \langle x^2y, xz^3\rangle$. we already know that the variety $\mathbf{V}(I_0) = \mathbf{V}(x) \cup \mathbf{V}(y,z) = \{[0:1:z], z \in k\} \cup \{[0:0:1]\} \cup \{[1:0:0]\}$ is of dimension 1. Something new arises with the higher dimension: since I_0 does not contain any monomial of the form $y^\beta z^\gamma$, we have a *plane* of exponents $\{0\} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ in $\mathcal{C}(I_0)$. We notice that it is also because I_0 does not contain any monomial of the form $y^\beta z^\gamma$ (or in other words, x divides any monomial in I_0) that we have that the one-dimensional linear variety $\mathbf{V}(x)$ is included in $\mathbf{V}(I_0)$. Let us conclude the study of $\mathcal{C}(I_0)$ before formalizing this link between the dimension of the variety and the structure of the set of monomials not the ideal.

Note that here none of the translates of the plane $\{i\} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ for $i > 0$ is in $\mathcal{C}(I_0)$ because $xz^3 \in I_0$, but obviously this could happen for other ideals. Overall, $\mathcal{C}(I_0)$ is the (non disjoint) union of the *coordinate* plane $\{0\} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, of the *coordinate* line $\mathbb{Z}_{\geqslant 0} \times \{0\} \times \{0\}$, as well as of the five line translates $\mathbb{Z}_{\geqslant 0} \times \{0\} \times \{i\}$ for $1 \leqslant i < 3$ and $\{1\} \times \mathbb{Z}_{\geqslant 0} \times \{i\}$ for $0 \leqslant i < 3$. In the general case, there could also be a finite number of additional points; one can for instance check this fact while describing $\mathcal{C}(I)$ for $I = \langle x^2y, xz^3, x^4y^3z^5\rangle$. These points can be seen as a finite number of translates of the origin point $(0,0,0) \in \mathbb{Z}^3_{\geqslant 0}$.

In the general case, we want to study $\mathcal{C}(I) = \{\alpha \in \mathbb{Z}^{n+1}_{\geqslant 0} \mid x^\alpha \notin I\}$ for a monomial ideal I in $k[x_0,\ldots,x_n]$. In the examples above, we have drawn two main conclusions in dimensions 2 and 3: $\mathcal{C}(I)$ is the union of the origin point, of what we called coordinate lines and planes, and of a finite number of translates of these three kinds of sets; and the dimension of $\mathbf{V}(I)$ seems linked to these sets composing $\mathcal{C}(I)$. We are now going to state these ideas more formally. We denote by $e_0 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$ the "canonical basis" of $\mathbb{Z}^{n+1}_{\geqslant 0}$ and for $0 \leqslant i_1 < \cdots < i_r \leqslant n$ we call $[e_{i_1},\ldots,e_{i_r}] = \{\sum_{1\leqslant j\leqslant r} a_j e_{i_j},\ a_j \in \mathbb{Z}_{\geqslant 0}\}$ an *r-dimensional coordinate subspace of* $\mathbb{Z}^{n+1}_{\geqslant 0}$. Its *translates* are the sets $\alpha + [e_{i_1},\ldots,e_{i_r}]$ where $\alpha \in \mathbb{Z}^{n+1}_{\geqslant 0}$ is orthogonal to $[e_{i_1},\ldots,e_{i_r}]$, that is, $\alpha = \sum_{i\notin\{i_1,\ldots,i_r\}} \alpha_i e_i$ for some $\alpha_i \in \mathbb{Z}_{\geqslant 0}$. We use the convention that when $r = 0$, $[e_{i_1},\ldots,e_{i_r}] = \{0\}$.

Proposition 9. *Let $I \subset k[x_0,\ldots,x_n]$ be a proper monomial ideal.*

1. *The projective coordinate subspace $\mathbf{V}(x_i,\ i \notin \{i_1,\ldots,i_r\})$ is contained in $\mathbf{V}(I)$ if and only if the coordinate subspace $[e_{i_1},\ldots,e_{i_r}]$ is contained in $\mathcal{C}(I)$.*

2. *If the largest coordinate subspace in $\mathcal{C}(I)$ is $\{0\}$ then $\mathbf{V}(I) = \emptyset$, otherwise $\mathbf{V}(I) \neq \emptyset$ and the largest coordinate subspace in $\mathcal{C}(I)$ has dimension $\dim \mathbf{V}(I) + 1$.*

Proof. Note that since I is proper we have at least the zero-dimensional coordinate subspace $\{0\}$ in $\mathcal{C}(I)$. *1.* $\Rightarrow$: let $\alpha \in [e_{i_1},\ldots,e_{i_r}]$, then x^α does not vanish at the point $[a_i] \in \mathbb{P}^n(k)$ with $a_{i_j} = 1$ for all j and $a_i = 0$ for $i \notin \{i_1,\ldots,i_r\}$, so $x^\alpha \notin I$, and thus $\alpha \in \mathcal{C}(I)$. $\Leftarrow$: every monomial in I is divisible by a variable x_i for $i \notin \{i_1,\ldots,i_r\}$ so it vanishes on the set $\mathbf{V}(x_i,\ i \notin \{i_1,\ldots,i_r\})$ of all points $[a_i] \in \mathbb{P}^n(k)$ which are such that $a_i = 0$ for all $i \notin \{i_1,\ldots,i_r\}$. *2.* follows from *1.* □

For consistency, we note that for a monomial ideal I (and k infinite), we always have $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$, which is a monomial ideal. This explains why the dimension of a variety V can be deduced from algebraic properties of I (through properties of $\mathcal{C}(I)$) for *any* monomial ideal I defining V.

Now that we have made explicit this link between the linear subspaces in $\mathbf{V}(I)$ and the coordinate subspaces in $\mathcal{C}(I)$, we conclude this section by a result on the structure of $\mathcal{C}(I)$. This will be the basis of another characterization of the dimension of a variety defined by a monomial ideal that we give in the next section. The proof of this result is by induction on the number of variables.

Proposition 10. *Let $I \subset k[x_0, \dots, x_n]$ be a proper monomial ideal. Then $\mathcal{C}(I)$ can be written as a finite union of translates of coordinate subspaces of $\mathbb{Z}_{\geqslant 0}^{n+1}$.*

2.3 Hilbert polynomial and function of a (monomial) ideal

In this section, we will characterize the dimension of a variety defined by a monomial ideal using the growth with respect to s of the number of monomials of total degree s that are not in the ideal. In Section 3, we will be able to use this characterization in order to define the dimension of any variety in $\mathbb{P}^n(k)$.

Let us first go back to the case of a monomial ideal I in $k[x, y]$. We want to count the number $\mathrm{HF}_I(s)$ (*Hilbert function*) of exponents of total degree s in $\mathcal{C}(I)$, in the four cases listed above (see Figure 1 for examples):

(i) $\mathcal{C}(I)$ is finite, so that for s large enough we have $\mathrm{HF}_I(s) = 0$.

(ii) No power of y is in I, so we can choose $i > 0$ minimal such that $x^i y^\beta \in I$ for some β. Then for s large enough we see that $\mathrm{HF}_I(s) = i$ is a constant with respect to s.

(iii) No power of x is in I; this case is similar to the previous one.

(iv) No power of x or y is in I, so that we can choose $i > 0$ minimal such that $x^i y^\beta \in I$ for some β and $j > 0$ minimal such that $x^\alpha y^j \in I$ for some α. Then for s large enough we see that $\mathrm{HF}_I(s) = i + j$ is a constant with respect to s.

Now we study the case of $I_0 = \langle x^2 y, xz^3 \rangle$ in $k[x, y, z]$. It is easy to verify that for $s \geqslant 4$, there are $s+1$ exponents of degree s in the plane $\mathbb{Z}_{\geqslant 0} \times \{0\} \times \{0\}$, and one exponent of degree s in each of the six lines $\mathbb{Z}_{\geqslant 0} \times \{0\} \times \{i\}$ and $\{1\} \times \mathbb{Z}_{\geqslant 0} \times \{i\}$ for $0 \leqslant i < 3$; no exponent was counted twice or more. Thus, in this example, $\mathrm{HF}_{I_0}(s) = s + 7$ for every $s \geqslant 4$. Defining the *Hilbert polynomial* $\mathrm{HP}_{I_0} = X + 7 \in \mathbb{Z}[X]$, it is of degree 1 and we have $\mathrm{HF}_{I_0}(s) = \mathrm{HP}_{I_0}(s)$ for every $s \geqslant 4$. The term of degree 1 in HP_{I_0} comes from the fact that we have a plane of exponents in $\mathcal{C}(I_0)$, which is also linked to $\mathbf{V}(I_0)$ having dimension 1 as we showed in Proposition 9. Having a second look at the four cases above concerning $k[x, y]$, we conclude that in all situations observed thus far, there is a polynomial $\mathrm{HP}_I(s)$ of degree $\dim \mathbf{V}(I)$ with integer values such that for all s sufficiently large we have $\mathrm{HF}_I(s) = \mathrm{HP}_I(s)$.

We would like to prove such a property for any monomial ideal I of $k[x_0, \dots, x_n]$. Recall that from Proposition 10 we know that $\mathcal{C}(I)$ is the union of coordinate subspaces and their translates, so we first need to count the number of exponents $\alpha \in \mathbb{Z}_{\geqslant 0}^{n+1}$ of total degree $|\alpha| = s$ that lie in such a subspace. It is known that the number of monomials in r variables of total degree s is $\binom{r-1+s}{s}$, thus the number of exponents of total degree s in an r-dimensional coordinate subspace is $\binom{r-1+s}{s}$. We deduce from this that the number of exponents of total degree s in a translate $\alpha_0 + [e_{i_1}, \dots, e_{i_r}]$ (where α_0 is orthogonal to $[e_{i_1}, \dots, e_{i_r}]$) is $\binom{r-1+s-|\alpha_0|}{s-|\alpha_0|}$, provided $s > |\alpha_0|$. Note that for fixed α_0, this binomial coefficient is a polynomial in s of degree $r-1$, which takes integer values at every integer $s > |\alpha_0|$ and whose leading coefficient is positive. Actually, any polynomial which takes integer values at sufficiently large integers can be written as an integer combination of such binomial coefficients.

With these remarks, one can prove the following theorem by induction, writing $\mathcal{C}(I)$ as a finite union of translates of coordinate subspaces of $\mathbb{Z}_{\geqslant 0}^{n+1}$, at least one of which is of dimension $\dim \mathbf{V}(I) + 1$, and finally using the Inclusion-Exclusion Principle.

Theorem 11. *Let I be a monomial ideal in $k[x_0, \ldots, x_n]$ and define the* Hilbert function $\mathrm{HF}_I(s)$ *as the number of monomials not in I of total degree s. Then, there is a polynomial* $\mathrm{HP}_I(s)$ *such that for all s sufficiently large,* $\mathrm{HF}_I(s) = \mathrm{HP}_I(s)$*.* $\mathrm{HP}_I(s)$ *is called the* Hilbert polynomial *of I.* $\mathrm{HP}_I(s) = 0$ *if and only if* $\mathbf{V}(I) = \emptyset$*, and if this is not the case,* $\mathrm{HP}_I(s)$ *has a positive leading coefficient and the degree of* $\deg \mathrm{HP}_I(s)$ *is the dimension of* $\mathbf{V}(I)$*.*

Furthermore, since $\mathrm{HP}_I(s)$ takes integer values for all sufficiently large integer s, we can write $\mathrm{HP}_I(s) = \sum_{i \leqslant d} b_i \binom{s}{d-i}$ for some integers $b_0, \ldots, b_d$ with $b_0 > 0$, where $d = \dim \mathbf{V}(I)$.

3 Dimension of varieties and properties

3.1 Dimension of a projective variety

We begin by defining the Hilbert function of any homogeneous ideal I in $k[x_0, \ldots, x_n]$. Counting the number of monomials of degree s not in I does not seem to be a helpful generalization of what we have exposed in Section 2. Choose for instance $I = \langle x_0^2 + x_1^2 \rangle$, then no monomial is in I, so that the set of exponents of monomials not in I is $\mathbb{Z}_{\geqslant 0}^{n+1}$; this does not seem to indicate anything about some notion of dimension of $\mathbf{V}(I)$. We have to take into account the fact that the two monomials x_0^2 and x_1^2 may not be in I but their sum is, which means they are somehow dependent when it comes to studying $\mathbf{V}(I)$: for example, if the monomial x_0^2 vanishes at a point of $\mathbf{V}(I)$, then the monomial x_1^2 also vanishes at this point.

Let $k[x_0, \ldots, x_n]_s$ denote the set of homogeneous polynomials of degree s together with the zero polynomial. This is a k-vector space of finite dimension $\dim_k k[x_0, \ldots, x_n]_s = \binom{n+s}{s}$, one basis of which is the set of monomials of degree s in $k[x_0, \ldots, x_n]$. Let also $I_s = I \cap k[x_0, \ldots, x_n]_s$ denote the set of polynomials of degree s in I; this is a vector subspace of $k[x_0, \ldots, x_n]_s$. What we did in Section 2 in the case of a monomial ideal I was counting the number of monomials $\mathrm{HF}_I(s)$ in the complement $k[x_0, \ldots, x_n]_s - I_s$, but as we have noted above, in the general case of a homogeneous ideal I this number is not helpful since for instance for many ideals I it will be $\dim_k k[x_0, \ldots, x_n]_s$. Our remark above concerning the dependency between monomials in $k[x_0, \ldots, x_n]_s - I_s$ leads to rather consider the quotient vector space $k[x_0, \ldots, x_n]_s / I_s$. We recall that this set is a k-vector space of dimension $\dim_k k[x_0, \ldots, x_n]_s - \dim_k I_s$. Note that in the case of a monomial ideal I, the equivalence classes modulo I_s of the monomials in $k[x_0, \ldots, x_n]_s - I_s$ form a basis of the k-vector space $k[x_0, \ldots, x_n]_s / I_s$, so that we have $\mathrm{HF}_I(s) = \dim_k k[x_0, \ldots, x_n]_s / I_s$. Considering the quotient instead of the set difference turns out to be the right point of view for defining the Hilbert function of any homogeneous ideal.

Definition 12. Let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. The *Hilbert function* of I is defined on positive integers by $\mathrm{HF}_I(s) = \dim_k k[x_0, \ldots, x_n]_s / I_s = \dim_k k[x_0, \ldots, x_n]_s - \dim_k I_s$.

Here is now the fundamental result that will allow us to link the Hilbert function of any homogeneous ideal to what we have studied in Section 2 about monomial ideals.

Theorem 13. *Let I be a homogeneous ideal and fix a monomial order on $k[x_0, \ldots, x_n]$. Then I and the monomial ideal $\langle \mathrm{LT}(I) \rangle$ have the same Hilbert function,* $\mathrm{HF}_I(s) = \mathrm{HF}_{\langle \mathrm{LT}(I) \rangle}(s)$*.*

Proof. For a fixed s, there are finitely many leading monomials of elements of I_s: we can write $\{\mathrm{LM}(f),\ f \in I_s\} = \{\mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_m)\}$ for some $f_i \in I_s$ with $\mathrm{LM}(f_1) > \cdots > \mathrm{LM}(f_m)$. Then, we observe that $f_1, \dots, f_m$ form a basis of I_s (as a k-vector space) and $\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_m)$ form a basis of $\langle \mathrm{LT}(I)\rangle_s$, so that I_s and $\langle \mathrm{LT}(I)\rangle_s$ have the same dimension, which conclude the proof.

Because of how we ordered the $\mathrm{LM}(f_i)$'s, any linear combination $a_j\mathrm{LM}(f_j) + \cdots + a_m\mathrm{LM}(f_m)$ with $a_j \neq 0$ cannot be zero, and the same holds for combinations $a_jf_j + \cdots + a_mf_m$ with $a_j \neq 0$. This proves that the two families are linearly independent over k.

Now, assume there exists $f \in I_s - \mathrm{Span}(f_1, \dots, f_m)$, and choose it with minimal leading monomial. We have $\mathrm{LM}(f) = \mathrm{LM}(f_j)$ for some j and $\mathrm{LT}(f) = \lambda\mathrm{LT}(f_j)$ for some $\lambda \in k$, so that $f - \lambda f_j$ is in I_s with a smaller leading monomial than f. So by minimality $f - \lambda f_j$ belongs to $\mathrm{Span}(f_1, \dots, f_m)$ and so does f, which is a contradiction; thus $I_s = \mathrm{Span}(f_1, \dots, f_m)$.

To see that $\langle \mathrm{LT}(I)\rangle_s$ is generated (as a k-vector space) by $\{\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_m)\}$, we note that $\langle \mathrm{LT}(I)\rangle_s$ is generated by $\{\mathrm{LM}(f),\ f \in I \text{ with } \mathrm{LM}(f) \text{ of total degree } s\}$ and we check that the latter generating set equals the former. Indeed, given $f \in I$ with $\mathrm{LM}(f)$ of total degree s, its homogeneous component h of degree s is in I, so actually $h \in I_s$. Besides $\mathrm{LM}(f) = \mathrm{LM}(h)$, so that $\mathrm{LM}(f) = \mathrm{LM}(f_i)$ for some i, which concludes the proof. □

A direct consequence of this theorem is that there is a polynomial called the *Hilbert polynomial* of I and denoted by $\mathrm{HP}_I(s)$ such that for s sufficiently large, $\mathrm{HF}_I(s) = \mathrm{HP}_I(s)$. Furthermore, one can write $\mathrm{HP}_I(s) = \sum_{i \leqslant d} b_i \binom{s}{d-i}$ for some integers $b_0, \dots, b_d$ with $b_0 > 0$ and $d = \dim \mathbf{V}(\langle \mathrm{LT}(I)\rangle)$ (or $\mathrm{HP}_I(s) = 0$ which happens exactly when $\mathbf{V}(\langle \mathrm{LT}(I)\rangle) = \emptyset$).

Having Section 2 in mind, one can hope to define the dimension of a variety thanks to the degree of some Hilbert polynomial. So far, we have seen that $\deg \mathrm{HP}_I(s)$ basically indicates how far I_s is from filling up the whole space $k[x_0, \dots, x_n]_s$; let us now show that $\deg \mathrm{HP}_I(s)$ bears some geometric properties of $\mathbf{V}(I)$. Indeed, the degree of the Hilbert polynomial is the same for a whole collection of ideals defining the same variety (namely all ideals having the same radical): for any homogeneous ideal I, we have $\deg \mathrm{HP}_I(s) = \deg \mathrm{HP}_{\sqrt{I}}(s)$. This is immediate for a monomial ideal I since in this case $\mathbf{V}(I) = \mathbf{V}(\sqrt{I})$ have the same dimension; from this we deduce the property for any homogeneous ideal I using the inclusion $\langle \mathrm{LT}(I)\rangle \subset \langle \mathrm{LT}(\sqrt{I})\rangle \subset \sqrt{\langle \mathrm{LT}(I)\rangle}$ and the fact that for two homogeneous ideals $I_1 \subset I_2$ in $k[x_0, \dots, x_n]$, we have $\deg \mathrm{HP}_{I_1}(s) \leqslant \deg \mathrm{HP}_{I_2}(s)$, which is easy by definition of the Hilbert polynomial. This leads us to the following definition.

Definition 14. The *dimension* of a variety $V \subset \mathbb{P}^n(k)$ is the degree of the Hilbert polynomial of the ideal of V, $\dim V = \deg \mathrm{HP}_{\mathbf{I}(V)}$.

As an example, we determine the dimension of a variety consisting of a single point, $V = \{p\}$ where $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n(k)$. Let i be the index of a nonzero coordinate $a_i \neq 0$ of p; then $V = \mathbf{V}(I)$ where $I = \langle a_ix_0 - a_0x_i, a_ix_1 - a_1x_i, \dots, a_ix_n - a_nx_i\rangle$. (We note that I is radical, so that if k is algebraically closed the Nullstellensatz tells us $I = \mathbf{I}(V)$; this actually still holds for any infinite field k but we will not need this result.) We choose a lexigraphic order of the monomials for which all variables are greater than x_i, for example $x_0 > \cdots > x_{i-1} > x_{i+1} > \cdots > x_n > x_i$. For all $j \neq i$ we have $\mathrm{LM}(a_ix_j - a_jx_i) = x_j$ so that $x_j \in \langle \mathrm{LT}(I)\rangle$. However, since $a_i \neq 0$ obviously $x_i \notin I$, which implies $x_i \notin \langle \mathrm{LT}(I)\rangle$ thanks to the monomial order we specified; thus $\langle \mathrm{LT}(I)\rangle = \langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\rangle$ (therefore, the generators which we have used to define I actually form a Gröbner basis of I for this monomial order). We have $I \subset \mathbf{I}(V)$, so that $\langle \mathrm{LT}(\mathbf{I}(V))\rangle$ contains $\langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\rangle$. Because of the monomial order we have chosen, if $x_i^r \in \langle \mathrm{LT}(\mathbf{I}(V))\rangle$ for some $r \geqslant 0$, then $x_i^r \in \mathbf{I}(V)$; since $\mathbf{I}(V)$ is radical we would have $x_i \in \mathbf{I}(V)$ and finally $\langle x_0, \dots, x_n\rangle \subset \mathbf{I}(V)$, which is a contradiction since V is nonempty. We have proven that $\langle \mathrm{LT}(\mathbf{I}(V))\rangle = \langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\rangle$. Now, the variety of $\langle \mathrm{LT}(\mathbf{I}(V))\rangle$ is the coordinate point

$\{[0 : \cdots : 0 : 1 : 0 : \cdots : 0]\}$ with nonzero coordinate at index i; it has dimension 0. Then, the degree of the Hilbert polynomial of $\langle \mathrm{LT}(\mathbf{I}(V))\rangle$ is 0 and according to Theorem 13, $\dim V = 0$.

Thanks to the correspondence between radical ideals and varieties when the base field is algebraically closed, the following result helps to effectively compute the dimension of a variety.

Theorem 15. *Assume k is algebraically closed. Let I be a homogeneous ideal of $k[x_0, \ldots, x_n]$ such that $V = \mathbf{V}(I)$ is nonempty. Then $\dim V = \deg \mathrm{HP}_I(s)$.*

Furthermore, for any monomial order we have $\dim V = \deg \mathrm{HP}_{\langle \mathrm{LT}(I)\rangle}(s)$ and this number is the maximal dimension of a projective coordinate subspace in $\mathbf{V}(\langle \mathrm{LT}(I)\rangle)$.

When k is algebraically closed, the emptiness of $\mathbf{V}(I) \subset \mathbb{P}^n(k)$ for a homogeneous ideal I is something we can effectively decide. Indeed, a corollary of the Nullstellensatz states that $\mathbf{V}(I)$ is empty if and only if, given some monomial order and $\mathcal{G}$ the reduced Gröbner basis of I, for each $0 \leqslant i \leqslant n$ there is $g \in \mathcal{G}$ such that $\mathrm{LT}(g)$ is a nonnegative power of x_i. We further remark that $\mathbf{V}(I)$ is empty if and only if $\mathrm{HP}_I(s) = 0$ is the zero polynomial.

When k is not algebraically closed, the second part of the theorem still holds as long as $I = \mathbf{I}(V)$. Thus, Theorem 15 gives us an algorithm to compute the dimension of the nonempty variety $V = \mathbf{V}(I)$ when k is algebraically closed or $I = \mathbf{I}(V)$: compute a Gröbner basis $\{g_1, \ldots, g_m\}$ for I using any monomial order, so that $\langle \mathrm{LT}(I)\rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m)\rangle$, and then compute the maximal dimension of a projective coordinate subspace in $\mathbf{V}(\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s)\rangle)$ using Algorithm 1. To see how it fails over $k = \mathbb{R}$ (which is not algebraically closed), one can study the ideal $I = \langle x^2 + y^2\rangle$ in $\mathbb{R}[x, y, z]$. The Hilbert polynomial of I, $\mathrm{HP}_I(s) = 2s + 1$, is of degree 1 while the variety $\mathbf{V}(I) = \mathbf{V}(x^2 + y^2) = \mathbf{V}(x, y) = \{[0 : 0 : 1]\}$ is (a projective coordinate subspace) of dimension 0.

Here is a first basic property of dimension, on any infinite field k. Let $V_1 \subset V_2$ be two varieties. Then $\mathbf{I}(V_2) \subset \mathbf{I}(V_1)$ so that $\mathbf{I}(V_2)_s$ is a vector subspace of $\mathbf{I}(V_1)_s$ for any s, which ensures that $\mathrm{HF}_{\mathbf{I}(V_1)}(s) \leqslant \mathrm{HF}_{\mathbf{I}(V_2)}(s)$. Looking at the degrees of the corresponding Hilbert polynomials, we easily deduce that $\dim V_1 \leqslant \dim V_2$.

3.2 Dimension and number of equations

In what follows, we assume k algebraically closed. In this section, we study some relationships between the number of polynomials defining a variety and the dimension of this variety. This extends well-known results in linear algebra such as the fact that a linear subspace of dimension $n-1$, called a *hyperplane*, can be defined as the set of solutions of one nontrivial linear equation.

Proposition 16. *Let $f \in k[x_0, \ldots, x_n]$ be a nonconstant homogeneous polynomial. Then, the dimension of the* hypersurface $\mathbf{V}(f) \subset \mathbb{P}^n(k)$ *is* $\dim \mathbf{V}(f) = n - 1$.

We note that this does not hold over $\mathbb{R}$, as a previous example shows: actually every variety $\mathbf{V}(f_1, \ldots, f_m)$ in $\mathbb{P}^n(\mathbb{R})$ where the f_i's are homogeneous with the same degree can be thought as the hypersurface $\mathbf{V}(f_1^2 + \cdots + f_m^2)$. Now, we study the same question for *subvarieties*, that is, replacing the ambient space $\mathbb{P}^n(k)$ by a variety $\mathbf{V}(I)$. This corresponds to adding an equation f to a predefined set of equations which generates I, and studying whether and how the dimension of the variety $\mathbf{V}(I + \langle f\rangle) = \mathbf{V}(I) \cap \mathbf{V}(f)$ changes.

Proposition 17. *Let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$ and let f be a nonconstant homogeneous polynomial. Then,* $\dim \mathbf{V}(I) \geqslant \dim \mathbf{V}(I + \langle f\rangle) \geqslant \dim \mathbf{V}(I) - 1$. *Furthermore, if the class of f in the quotient ring $k[x_0, \ldots, x_n]/I$ is not a zero divisor, then* $\dim \mathbf{V}(I + \langle f\rangle) = \dim \mathbf{V}(I) - 1$.

Proof. As it has been remarked before, $\mathbf{V}(I + \langle f \rangle) \subset \mathbf{V}(I)$ implies $\dim \mathbf{V}(I) \geqslant \dim \mathbf{V}(I + \langle f \rangle)$. Now, we will conclude the proof by comparing $\mathrm{HP}_{I+\langle f \rangle}(s)$ to a polynomial of degree $d-1$. Assume f has total degree r and fix $s \geqslant r$. Then it is easily checked that the natural surjection $\pi : k[x_0, \dots, x_n]_s / I_s \longrightarrow k[x_0, \dots, x_n]_s/(I + \langle f \rangle)_s$ has kernel $\ker \pi = \{fh,\, h \in k[x_0, \dots, x_n]_{s-r}/I_{s-r}\}$, whose dimension is $\dim \ker \pi \leqslant \dim k[x_0, \dots, x_n]_{s-r}/I_{s-r} = \mathrm{HF}_I(s-r)$. Using Hilbert functions, the dimension theorem for linear maps gives us $\mathrm{HF}_I(s) \leqslant \mathrm{HF}_I(s-r) + \mathrm{HF}_{I+\langle f \rangle}(s)$. Thus, for all s sufficiently large we have $\mathrm{HP}_{I+\langle f \rangle}(s) \geqslant \mathrm{HP}_I(s) - \mathrm{HP}_I(s-r)$.

Now, let $d = \dim \mathbf{V}(I)$. The polynomial on the right-hand side of this inequality is of degree $d-1$ (in particular it is nonzero if $d = 1$, this case will be used for Proposition 19), so that $\deg \mathrm{HP}_{I+\langle f \rangle}(s)$ is of degree at least $d-1$. Furthermore, if the class of f in the quotient ring $k[x_0, \dots, x_n]/I$ is not a zero divisor then we have equality $\dim \ker \pi = \dim k[x_0, \dots, x_n]_{s-r}/I_{s-r} = \mathrm{HF}_I(s-r)$, and all inequalities above in the proof become equalities; then $\mathrm{HP}_{I+\langle f \rangle}(s)$ is of degree exactly $d-1$. ☐

By induction, we deduce that for r homogeneous polynomials $f_1, \dots, f_r$, we have $\dim \mathbf{V}(I) \geqslant \dim \mathbf{V}(I + \langle f_1, \dots, f_r \rangle) \geqslant \dim \mathbf{V}(I) - r$. We can also derive a more precise statement for the equality case, under some assumption on the f_i's.

Proposition 18. *Let $f_1, \dots, f_r$ be nonconstant homogeneous polynomials, and assume that for every $i \in \{1, \dots, r-1\}$, the class of f_{i+1} in the quotient ring $k[x_0, \dots, x_n]/\langle f_1, \dots, f_i \rangle$ is not a zero divisor. Then $\dim \mathbf{V}(f_1, \dots, f_r) = n - r$.*

Such a sequence $f_1, \dots, f_r$ is called a *regular sequence*; obviously, given a homogeneous ideal I, a regular sequence which generates I directly gives the dimension of $\mathbf{V}(I)$, but unfortunately not all homogeneous ideals I admit such a sequence (those who do are called *complete intersections*).

Note that in Proposition 17, when $\dim \mathbf{V}(I) \geqslant 1$, then $\mathrm{HP}_{I+\langle f \rangle}(s)$ is not the zero polynomial, so that $\mathbf{V}(I + \langle f \rangle)$ cannot be empty. This proves the following corollary, which states that a positive-dimensional variety meets every hypersurface in $\mathbb{P}^n(k)$.

Proposition 19. *Let $V \subset \mathbb{P}^n(k)$ be a variety of positive dimension. Then, for every nonconstant homogeneous polynomial $f \in k[x_0, \dots, x_n]$, $V \cap \mathbf{V}(f) \neq \emptyset$.*

A natural question is to ask what varieties of dimension 0 look like. All the ones we have encountered in the examples consisted of finitely many of points. In the next section, we will prove that this is always the case; in this proof we use the idea of decomposing a variety into the union of "simpler" varieties, which will lead us to focus on the so-called *irreducible* varieties of $\mathbb{P}^n(k)$.

3.3 Dimension and irreducible varieties

To introduce this section, we will describe zero-dimensional varieties. Unless specified, we do not make the assumption that k is algebraically closed.

Proposition 20. *Let $V \subset \mathbb{P}^n(k)$ be a nonempty variety. V is of dimension 0 if and only if it consists of finitely many points.*

Proof. First, we assume that $\dim V = 0$. For $0 \leqslant i \leqslant n$ we will show that there can be only finitely many points in $V \cap H_i$, where H_i is the set of points in $\mathbb{P}^n(k)$ whose coordinate at index i is nonzero; since $\mathbb{P}^n(k) = H_0 \cup \cdots \cup H_n$, this proves that $V = (V \cap H_0) \cup \cdots \cup (V \cap H_n)$ consists of finitely many points. Since $\dim V = 0$, the Hilbert polynomial of $I = \mathbf{I}(V)$ is a constant $c > 0$:

for all s sufficiently large, $\dim k[x_0,\dots,x_n]_s/I_s = c$. Fix s sufficiently large with also $s \geqslant c$, and fix $0 \leqslant i \leqslant n$. Let $p \in V \cap H_i$ and write $p = [a_0 : \cdots : a_n]$ with $a_i = 1$; we will show that all a_j's with $j \neq i$ can take only finitely many values. Let j be an index, $j \neq i$. The $s+1 > c$ classes modulo I_s of the degree s monomials $x_i^s, x_i^{s-1}x_j, \dots, x_i x_j^{s-1}, x_j^s$ are linearly dependent, which means that there exist $\lambda_0,\dots,\lambda_s \in k$ not all zero such that $\sum_{0\leqslant r\leqslant s} \lambda_r x_i^{s-r} x_j^r \in I_s$. In particular, this polynomial vanishes at every point of V. Then $\sum_{0\leqslant r\leqslant s} \lambda_r a_j^r = 0$, that is, a_j is a root of a nonzero univariate polynomial over k: it can take only finitely many values. Thus, each $V \cap H_i$ consists of finitely many points, and so do V.

Now, we assume that V consists of finitely many points and we want to show $\dim V = 0$. We have already proven above that $\dim V = 0$ when V consists of a single point. Intuitively, it seems that a finite union of points (which each has each dimension 0) cannot have a positive dimension; this intuition is further backed by the fact that if these finitely many points formed a positive dimensional variety, according to Proposition 19 they would have to meet every hypersurface of $\mathbb{P}^n(k)$. We will prove the more general Proposition 21 below, which concludes the proof of Proposition 20. □

Proposition 21. *If $V, W \in \mathbb{P}^n(k)$ are two varieties then* $\dim V \cup W = \max(\dim V, \dim W)$.

Proof. Let $I = \mathbf{I}(V)$ and $J = \mathbf{I}(W)$. Since $\mathbf{I}(V \cup W) = I \cap J$, we have $\dim V = \deg \mathrm{HP}_I(s)$, $\dim W = \deg \mathrm{HP}_J(s)$ and $\dim V \cup W = \deg \mathrm{HP}_{I\cap J}(s)$. It will be more convenient to work with the product ideal rather than the intersection, because we are going to focus on the ideals generated by the leading terms and the product IJ satisfies $\langle \mathrm{LT}(IJ)\rangle = \langle \mathrm{LT}(I)\rangle\langle \mathrm{LT}(J)\rangle$. (Although $\mathbf{V}(IJ) = V \cup W$, we cannot directly deduce $\dim V \cup W = \deg \mathrm{HP}_{IJ}(s)$ since we have not assumed k to be algebraically closed.) We have $IJ \subset I \cap J \subset \sqrt{IJ}$ so that $\deg \mathrm{HP}_{\sqrt{IJ}}(s) \leqslant \deg \mathrm{HP}_{I\cap J}(s) \leqslant \deg \mathrm{HP}_{IJ}(s)$, and we have seen before that the two outer terms are equal. Thus, we have $\dim V \cup W = \deg \mathrm{HP}_{IJ}(s)$.

Now, from $\langle \mathrm{LT}(IJ)\rangle = \langle \mathrm{LT}(I)\rangle\langle \mathrm{LT}(J)\rangle$ we get $\mathbf{V}(\langle \mathrm{LT}(IJ)\rangle) = \mathbf{V}(\langle \mathrm{LT}(I)\rangle) \cup \mathbf{V}(\langle \mathrm{LT}(J)\rangle)$. Let us finally write those varieties of monomial ideals as unions $\mathbf{V}(\langle \mathrm{LT}(I)\rangle) = V_1 \cup \cdots \cup V_v$ and $\mathbf{V}(\langle \mathrm{LT}(J)\rangle) = W_1 \cup \cdots \cup W_w$ of projective coordinate subspaces of $\mathbb{P}^n(k)$. Since $\mathbf{V}(\langle \mathrm{LT}(IJ)\rangle) = V_1 \cup \cdots \cup V_v \cup W_1 \cup \cdots \cup W_w$ is the variety of a monomial ideal written as a finite union of projective coordinate subspaces, its dimension is the maximum of the dimensions of these subspaces. In particular, $\dim \mathbf{V}(\langle \mathrm{LT}(IJ)\rangle) = \max(\dim \mathbf{V}(\langle \mathrm{LT}(I)\rangle), \dim \mathbf{V}(\langle \mathrm{LT}(J)\rangle))$. Going back to degrees of Hilbert polynomials, we have proven that $\deg \mathrm{HP}_{IJ}(s) = \max(\deg \mathrm{HP}_I(s), \deg \mathrm{HP}_J(s))$, hence the conclusion. □

Obviously, being able to decompose a variety into a finite union of varieties which are somehow "more fundamental" can provide easier proofs, better understanding or can help some computations. A special kind of varieties are those which cannot be written as the union of two strictly smaller varieties. A variety V is said to be *irreducible* if, whenever we write V as the union of two varieties $V = V_1 \cup V_2$, then either $V_1 = V$ or $V_2 = V$. For instance, we have seen that a zero-dimensional variety is irreducible if and only if it consists of a single point; we note that otherwise it is a finite union of irreducible (zero-dimensional) varieties. Besides, there is such a decomposition for any variety in $\mathbb{P}^n(k)$

Proposition 22. *Let $V \subset \mathbb{P}^n(k)$ be a variety. Then there exist irreducible varieties $V_1,\dots,V_m$ such that $V = V_1 \cup \cdots \cup V_m$ and $V_i \not\subset V_j$ for $i \neq j$. This decomposition is unique (up to the order of the terms) and it is called the* minimal decomposition *of the variety into irreducibles.*

One can prove this by contradiction, using the ascending chain condition for ideals in $k[x_0,\dots,x_n]$, which ensures that any descending chain of varieties eventually stabilizes.

Another example of irreducible varieties that we have already encountered came up in the decomposition of the variety of a monomial ideal as the union of finitely many projective coordinate subspaces. Still, how to prove that a coordinate subspace is an irreducible variety? The simplest method may be to study the ideal of this variety.

Proposition 23. *Let $V \subset \mathbb{P}^n(k)$ be a variety. Then V is irreducible if and only if $\mathbf{I}(V)$ is a prime ideal, that is, $k[x_0, \ldots, x_n]/\mathbf{I}(V)$ is an integral domain.*

Indeed, for a coordinate subspace $V = \mathbf{V}(x_{i_1}, \ldots, x_{i_t})$, we have $\mathbf{I}(V) = \langle x_{i_1}, \ldots, x_{i_t} \rangle$ and $k[x_0, \ldots, x_n]/\mathbf{I}(V)$ is (ring-)isomorphic to $k[x_0, \ldots, x_{n-t}]$ which is an integral domain.

We note that a prime ideal is always a radical ideal. Thus the functions $\mathbf{I}$ and $\mathbf{V}$ induce a one-to-one correspondence between homogeneous *prime* ideals properly contained in $\langle x_0, \ldots, x_n \rangle$ and nonempty *irreducible* varieties in $\mathbb{P}^n(k)$. As a corollary of Proposition 21, we have that the dimension of a variety is the largest of the dimensions of its irreducible components; this somehow generalizes our definition of the dimension of a variety of a monomial ideal.

The following result shows that when the ambient space is an irreducible variety V, adding an equation f which is independent from the ones defining V always makes the dimension exactly decrease by 1, unlike in the general case studied in Proposition 17.

Proposition 24. *Let k be an algebraically closed field.*

Let I be a homogeneous prime ideal in $k[x_0, \ldots, x_n]$ and let f be a nonconstant homogeneous polynomial not in I. Then $\dim \mathbf{V}(I + \langle f \rangle) = \dim \mathbf{V}(I) - 1$.

This is a direct consequence of Proposition 17 since the class of f in the integral domain $k[x_0, \ldots, x_n]/I$ is nonzero, thus not a zero divisor. As a corollary, if $W \subset V$ are two varieties with $W \neq V$, then $\dim W < \dim V$.

3.4 Dimension and algebraic independence

We have defined the dimension of a variety V using the degree of the Hilbert polynomial of the ideal of V; this degree somehow measures how fast with s the vector spaces $k[x_0, \ldots, x_n]_s/\mathbf{I}(V)_s$ grow inside the quotient ring $k[x_0, \ldots, x_n]/\mathbf{I}(V)$. We first focused on these quotient spaces in order to account for some relations between the monomials with respect to the variety: when studying $\mathbf{V}(x_0^2 + x_1^2)$, the monomials x_0^2 and x_1^2 seem dependent in some way. In particular, their classes in $k[x_0, \ldots, x_n]/\mathbf{I}(V)$ only differ by a constant factor, $x_0^2 = -x_1^2$.

Definition 25. Elements $\Phi_1, \ldots, \Phi_r \in k[x_0, \ldots, x_n]/\mathbf{I}(V)$ are said to be *algebraically independent* if there is no nonzero polynomial $f \in k[y_1, \ldots, y_r]$ such that $f(\Phi_1, \ldots, \Phi_r) = 0$ in $k[x_0, \ldots, x_n]/\mathbf{I}(V)$.

In the example above, x_0^2 and x_1^2 are algebraically dependent since $f = y_1 + y_2 \in k[y_1, y_2]$ satisfies $f(x_0^2, x_1^2) = 0$. The elements x_0 and x_1 are also algebraically dependent, a relation being given by the polynomial $f = y_1^2 + y_2^2$. Another example is when $\mathbf{I}(V) = \{0\}$, that is, $V = \mathbb{P}^n(k)$. Then, $x_0, \ldots, x_n$ are algebraically independent since $k[x_0, \ldots, x_n]/\mathbf{I}(V) = k[x_0, \ldots, x_n]$ and a polynomial $f \in k[y_0, \ldots, y_n]$ satisfying $f(x_0, \ldots, x_n) = 0 \in k[x_0, \ldots, x_n]$ must be the zero polynomial. Going back to the general case, obviously, if some elements $\Phi_1, \ldots, \Phi_r \in k[x_0, \ldots, x_n]/\mathbf{I}(V)$ are algebraically independent, then the Φ_i's are distinct and nonzero, and any subset of these elements $\Phi_{i_1}, \ldots, \Phi_{i_s}$ are still algebraically independent. Now, we link this notion of independence with the dimension of V.

Theorem 26. *Let $V \subset \mathbb{P}^n(k)$ be a projective variety. Then, the maximal number of elements which are algebraically independent in $k[x_0,\dots,x_n]/\mathbf{I}(V)$ is $\dim V + 1$.*

Proof. We first exhibit $\dim V + 1$ algebraically independent elements in $k[x_0,\dots,x_n]/\mathbf{I}(V)$ and then we prove that there cannot be strictly more than $\dim V + 1$.

Let $d = \dim V$ and $I = \mathbf{I}(V)$. Having fixed a monomial order on $k[x_0,\dots,x_n]$, d is the maximal dimension of a projective coordinate subspace $W \subset \mathbf{V}(\langle \mathrm{LT}(I)\rangle)$. Such a subspace can be written $W = \mathbf{V}(x_i, i \notin \{i_1,\dots,i_{d+1}\})$ for some $0 \leqslant i_1 < \cdots < i_{d+1} \leqslant n$. Since $W \subset \mathbf{V}(\langle \mathrm{LT}(I)\rangle)$, according to Proposition 9 no monomial in $\langle \mathrm{LT}(I)\rangle$ can involve only variables among $x_{i_1},\dots,x_{i_{d+1}}$. Then, no polynomial in I can involve only variables among $x_{i_1},\dots,x_{i_{d+1}}$, otherwise its leading monomial would be a monomial in $\langle \mathrm{LT}(I)\rangle$ involving only these variables; that is,

$$I \cap k[x_{i_1},\dots,x_{i_{d+1}}] = \{0\}.$$

Now, if $f \in k[y_1,\dots,y_{d+1}]$ is a polynomial such that $f(x_{i_1},\dots,x_{i_{d+1}}) = 0$ in $k[x_0,\dots,x_n]/I$, the expression $f(x_{i_1},\dots,x_{i_{d+1}})$ is a polynomial in $I \cap k[x_{i_1},\dots,x_{i_{d+1}}] = \{0\}$, so f is the zero polynomial. Thus, the $d+1$ elements $x_{i_1},\dots,x_{i_{d+1}}$ are algebraically independent.

Now, we assume that we have r polynomials $f_1,\dots,f_r \in k[x_0,\dots,x_n]$ whose classes $\Phi_1,\dots,\Phi_r$ in $k[x_0,\dots,x_n]/I$ are algebraically independent. Let D denote the largest total degree of the f_i's; then for a polynomial $f \in k[y_1,\dots,y_r]$ of total degree at most s, $f(f_1,\dots,f_r)$ has total degree at most Ds. For every s, we denote by $k[x_0,\dots,x_n]_{\leqslant s}$ the vector space of polynomials of total degree at most s and $I_{\leqslant s} = I \cap k[x_0,\dots,x_n]_{\leqslant s}$. Then, for $f \in k[y_1,\dots,y_r]_{\leqslant s}$, $f(\Phi_1,\dots,\Phi_r) \in k[x_0,\dots,x_n]_{\leqslant Ds}/I_{\leqslant Ds}$. Since $\Phi_1,\dots,\Phi_r$ are algebraically independent, the linear map $\pi : k[y_1,\dots,y_r]_{\leqslant s} \longrightarrow k[x_0,\dots,x_n]_{\leqslant Ds}/I_{\leqslant Ds}$, $f \mapsto f(\Phi_1,\dots,\Phi_r)$ is one-to-one. Therefore, the dimension of the image of π is equal to $\dim k[y_1,\dots,y_r]_{\leqslant s} = \binom{s+r}{s}$, and it is at most $\dim k[x_0,\dots,x_n]_{\leqslant Ds}/I_{\leqslant Ds}$. Now, since every polynomial is uniquely written as the sum of its homogeneous components, $k[x_0,\dots,x_n]_{\leqslant Ds}$ is the direct sum of the subspaces $k[x_0,\dots,x_n]_t$ for $0 \leqslant t \leqslant Ds$, and $I_{\leqslant Ds}$ is the direct sum of I_t for $0 \leqslant t \leqslant Ds$. Thus,

$$\begin{aligned}
\dim k[x_0,\dots,x_n]_{\leqslant Ds}/I_{\leqslant Ds} &= \dim k[x_0,\dots,x_n]_{\leqslant Ds} \; - \; \dim I_{\leqslant Ds} \\
&= \sum_{0\leqslant t\leqslant Ds} \dim k[x_0,\dots,x_n]_t \; - \sum_{0\leqslant t\leqslant Ds} \dim I_t \\
&= \sum_{0\leqslant t\leqslant Ds} (\dim k[x_0,\dots,x_n]_t - \dim I_t) \\
&= \sum_{0\leqslant t\leqslant Ds} \dim k[x_0,\dots,x_n]_t/I_t \; = \sum_{0\leqslant t\leqslant Ds} \mathrm{HF}_I(t).
\end{aligned}$$

Let t_0 be an integer such that $\mathrm{HF}_I(t) = \mathrm{HP}_I(t)$ for $t \geqslant t_0$. Then writing $C = \sum_{0\leqslant t<t_0} \mathrm{HF}_I(t)$, we have $\dim k[x_0,\dots,x_n]_{\leqslant Ds}/I_{\leqslant Ds} = C + \sum_{t_0\leqslant t\leqslant Ds} \mathrm{HP}_I(t)$, which is a polynomial in s of degree $\deg \mathrm{HP}_I(s) + 1 = \dim V + 1$ with positive leading coefficient. Furthermore, we have proven that this polynomial is lower bounded by $\binom{s+r}{s}$ for all s, which is itself a polynomial in s of degree r with positive leading coefficient. This ensures that $r \leqslant \dim V + 1$. □

Looking at the proof more closely, we can refine the statement of the Theorem as follows.

Corollary 27. *The dimension of a variety V is the largest integer d for which there exist $d+1$ variables $x_{i_1},\dots,x_{i_{d+1}}$ such that $\mathbf{I}(V) \cap k[x_{i_1},\dots,x_{i_{d+1}}] = 0$.*