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Faster change of order algorithm for Gröbner bases under shape and stability assumptions

ACA 2022
Gebze Technical University, Turkey
15th August 2022

outline

- problem and result
- previous work
- new ingredients
- conclusion | perspectives

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- **problem and result**
- **previous work**
- **new ingredients**
- **conclusion | perspectives**

- problem: change of monomial order
- faster algorithm via univariate matrices
- improved software performances

problem: change of monomial order

Input:

- two monomial orders $\preccurlyeq_1$ and $\preccurlyeq_2$ on $\mathbb{K}[x_1, \ldots, x_n]$
- a reduced $\preccurlyeq_1$-Gröbner basis $\mathcal{G}$

Assumption:

- the ideal $\mathcal{I} = \langle \mathcal{G} \rangle$ is zero-dimensional

Output:

- the reduced $\preccurlyeq_2$-Gröbner basis of $\mathcal{I}$

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example: solving multivariate polynomial systems

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ f_m(x_1, \ldots, x_n) = 0 \end{cases}$$ with finitely many solutions over $\overline{\mathbb{K}}$

$\mathcal{G}_{\text{drl}}$ the reduced $\preccurlyeq_{\text{drl}}$-GB of $\langle f_1, \ldots, f_m \rangle$ [Faugère's F4/F5, 2002]

$\Big\downarrow$ change of order $\preccurlyeq_{\text{drl}} \rightarrow \preccurlyeq_{\text{lex}}$

$\mathcal{G}_{\text{lex}}$ the reduced $\preccurlyeq_{\text{lex}}$-GB of $\langle f_1, \ldots, f_m \rangle$

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example: multivariate interpolation with degree constraints

$\mathcal{I}$ = vanishing ideal of known points $\alpha_1, \ldots, \alpha_D \in \mathbb{K}^n$

$\preccurlyeq$ = monomial order defined from the degree constraints

$\mathcal{G}_{\text{lex}}$ the reduced $\preccurlyeq_{\text{lex}}$-GB of $\mathcal{I}$

[Möller-Bucherberger 1982, Cerlienco-Mureddu 1995, Ceria-Mora 2019]

$\downarrow$ change of order $\preccurlyeq_{\text{lex}} \rightarrow \preccurlyeq$

$\mathcal{G}$ the reduced $\preccurlyeq$-GB of $\mathcal{I}$

[Jérémy Berthomieu & Vincent Neiger & Mohab Safey El Din, ISSAC 2022]

deterministic change of order with complexity $\tilde{O}(t^{\omega-1}D)$

change of order: better complexity & faster implementation

for $\preccurlyeq_2 = \preccurlyeq_{\text{lex}}$, under classical assumptions (stability + shape position)

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description of complexity

- ω = complexity exponent of matrix multiplication
  $\tilde{O}(\cdot)$ hides a few logarithmic terms in $\frac{D}{t}$
- D = degree of the ideal $\mathcal{I} = \langle \mathcal{G} \rangle$
  $=$ vector space dimension of $\mathbb{K}[x]/\mathcal{I}$
- t = number of polynomials in $\mathcal{G}$ with leading term divisible by x_n
  (in particular, $t \leqslant D$)

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summary of previous results

general algorithms (deterministic, $\preccurlyeq_1 \to \preccurlyeq_2$):

- no assumption: $O(nD^3)$ [Faugere-Gianni-Lazard-Mora 1993]
- with stability: $O(nD^\omega \log(D))$ [Neiger-Schost 2020]

specific algorithms (randomized, $\preccurlyeq_{\text{drl}} \to \preccurlyeq_{\text{lex}}$, with stability+shape):

- dense linear algebra: $O(D^\omega \log(D))$ [Faugère-Gaudry-Huot-Renault 2014]
- sparse linear algebra: $O(tD^2)$ [Faugère-Mou 2011+2017]

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ingredients of new algorithm

- paradigm shift concerning the core computational object:

  $M \in \mathbb{K}^{D\times D}$ with t dense rows $\xrightarrow{\text{compress}}$ $P \in \mathbb{K}[x_n]^{t\times t}$ of degree $\frac{D}{t}$

  multiplication by x_n in $\mathbb{K}[x]/\mathcal{I}$ $\longrightarrow$ $\mathbb{K}[x_n]$-module, generates $\mathcal{I}$

- preserving essential consequence of stability: P obtained for free from $\mathcal{G}$

- new result: Hermite normal form of P yields $\mathcal{G}_{\text{lex}}$

```
sage: M.degree_matrix(shifts=[-1,2], row_wise=False)
[ 0 -2 -1]
[ 5 -2 -2]
```

hermite_form(*include_zero_rows=True, transformation=False*)

Return the Hermite form of this matrix.

The Hermite form is also normalized, i.e., the pivot polynomials are monic.

INPUT:

- include_zero_rows – boolean (default: True); if False, the zero rows in the output deleted
- transformation – boolean (default: False); if True, return the transformation mat

software performance

EXAMPLES:

```
sage: M.<x> = GF(7)[]
sage: A = matrix(M, 2, 3, [x, 1, 2*x, x, 1+x, 2])
sage: A.hermite_form()
[      x       1     2*x]
[      0       x 5*x + 2]
sage: A.hermite_form(transformation=True)
(
[      x       1     2*x]  [1 0]
[      0       x 5*x + 2], [6 1]
)
sage: A = matrix(M, 2, 3, [x, 1, 2*x, 2*x, 2, 4*x])
sage: A.hermite_form(transformation=True, include_zero_rows=False)
([  x   1 2*x], [0 4])
sage: H, U = A.hermite_form(transformation=True, include_zero_rows=True); H, U
(
[  x   1 2*x]  [0 4]
[  0   0   0], [5 1]
)
sage: U * A == H
True
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sage: U * A
[  x   1 2*x]
sage: U * A == H
True
```

See also: is_hermite() .

is_hermite(*row_wise=True, lower_echelon=False, include_zero_vectors=True*)

Return a boolean indicating whether this matrix is in Hermite form.

```
164     // order that remains to be dealt with
165     VecLong rem_order(order);
166
167     // indices of columns/orders that remain to be dealt with
168     VecLong rem_index(cdim);
169     std::iota(rem_index.begin(), rem_index.end(), 0);
170
171     // all along the algorithm, shift = shifted row degrees of approximant ba
172     // (initially, input shift = shifted row degree of the identity matrix)
173
174     while (not rem_order.empty())
175     {
176         /** Invariant:
177          *  - appbas is a shift-ordered weak Popov approximant basis for
178          *  (pmat,reached_order) where doneorder is the tuple such that
179          *  -->reached_order[j] + rem_order[j] == order[j] for j appearing in
180          *  -->reached_order[j] == order[j] for j not appearing in rem_index
185         long j=0; // value if columnwise (order_wise==False)
186         if (order_wise)
187             j = std::distance(rem_order.begin(), std::max_element(rem_order.b
    );
188
189         long deg = order[rem_index[j]] - rem_order[j];
190
191         // record the coefficients of degree deg of the column j of residual
192         // also keep track of which of these are nonzero,
193         // and among the nonzero ones, which is the first with smallest shift
194         Vec<zz_p> const_residual;
195         const_residual.SetLength(rdim);
196         VecLong indices_nonzero;
197         long piv = -1;
198         for (long i = 0; i < rdim; ++i)
199         {
200             const_residual[i] = coeff(residual[i][j],deg);
201             if (const_residual[i] != 0)
202             {
203                 indices_nonzero.push_back(i);
204                 if (piv<0 || shift[i] < shift[piv])
205                     piv = i;
206             }
207         }
208
209         // if indices_nonzero is empty, const_residual is already zero, there
210         if (not indices_nonzero.empty())
211         {
212             // update all rows of appbas and residual in indices_nonzero exce
```

src/mat_lzz_pX_approximant.cpp

open-source C/C++ software libraries

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multivariate polynomial systems

msolve `https://msolve.lip6.fr/`

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See also: `is_hermite()`.

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compared algorithms:

- sparse FGLM [Faugère-Mou 2011,2017]
- block-Wiedemann variant [folklore]
- new Hermite normal form-based algorithm
(without SIMD vectorization for the moment)

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software performance

random square system, n variables, degree d
over $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$ with 30-bit modulus p

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n	d	D	t	spFGLM	block	**HNF**
12	2	4096	924	6.5	5.5	**5.3**
14	2	16384	3432	1011	358	**240**
16	2	65536	12870	58744	22059	**11359**
8	3	6561	1107	23.6	18.7	**15.1**
9	3	19683	3139	1302	525	**314**
10	3	59049	8953	34844	13315	**6709**
6	4	4096	580	4	3.5	**3.5**
7	4	16384	2128	575	225	**157**
8	4	65536	8092	36454	13609	**7231**

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- **new ingredients**
- **conclusion | perspectives**

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  - stability and multiplication matrix
  - shape position and lexicographic basis
  - previously: dense or sparse linear algebra
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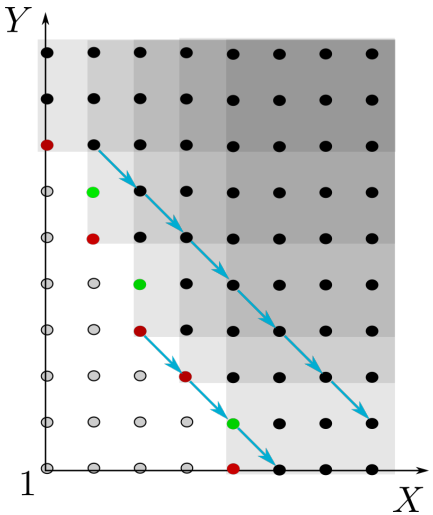
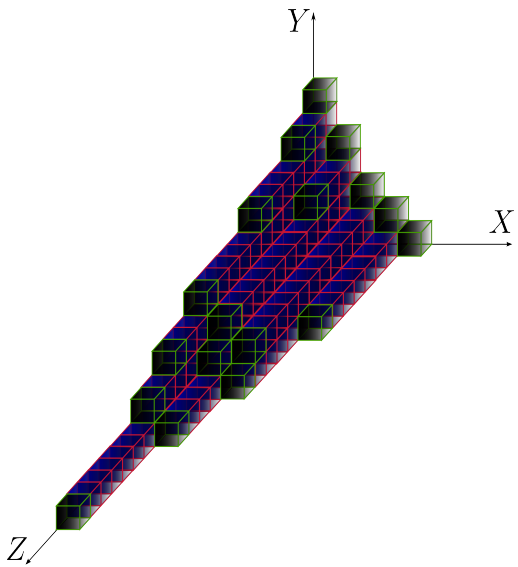
stability and multiplication matrix

x_n-stability: for any monomial $\mu \in \mathrm{lt}_\preccurlyeq(\mathcal{I})$ such that x_n divides μ, $\frac{x_i}{x_n}\mu \in \mathrm{lt}_\preccurlyeq(\mathcal{I})$ for all $i \in \{1,\dots,n-1\}$





- related to classical notions of stability and of Borel-fixedness [Herzog-Hibi 2011, Galligo 1974, Bayer-Stillman 1987]
- easily verified: considering $\mu = \mathrm{lt}_\preccurlyeq(g)$ for $g \in \mathcal{G}$ is sufficient

$\preccurlyeq$-monomial basis $\mathcal{B} = \{\varepsilon_1,\dots,\varepsilon_D\}$
$=$ monomials not in $\mathrm{lt}_\preccurlyeq(\mathcal{I})$
$=$ vector space basis of $\mathbb{K}[x]/\mathcal{I}$

- x_n-stability $\Leftrightarrow$ multiplying element $\varepsilon \in \mathcal{B}$ by x_n gives either $x_n\varepsilon \in \mathcal{B}$ or $x_n\varepsilon = \mathrm{lt}_\preccurlyeq(g)$ for some $g \in \mathcal{G}$
- in $\mathbb{K}[x]/\mathcal{I}$, the representation of $\mathrm{lt}_\preccurlyeq(g)$ on $\mathcal{B}$ is $\mathrm{lt}_\preccurlyeq(g) - g$

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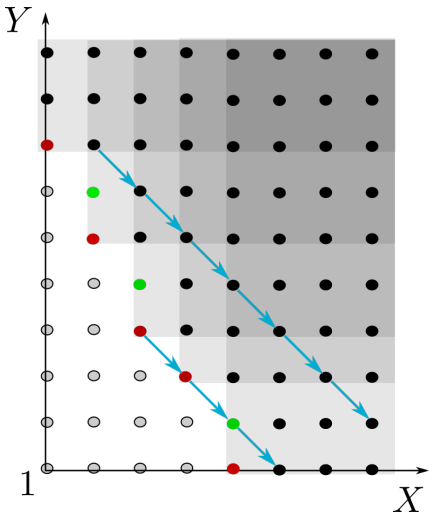

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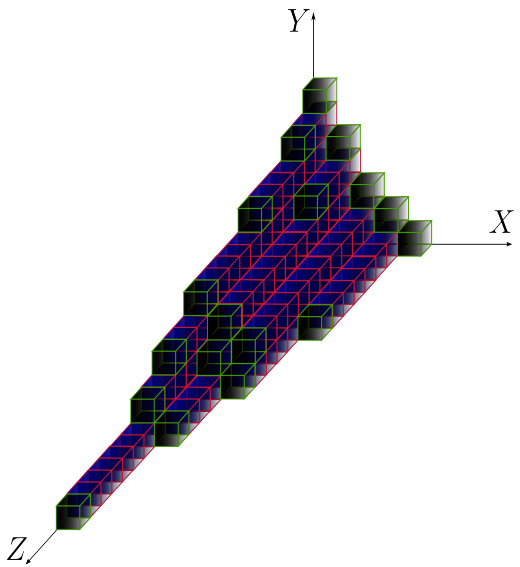

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multiplication matrix $M_n \in \mathbb{K}^{D\times D}$ of x_n in $\mathbb{K}[x]/\mathcal{I}$
- row i = representation of $x_n\varepsilon_i$ on $\mathcal{B}$
- deduced directly from $\hat{\mathcal{G}} = \{g \in \mathcal{G} \mid x_n \text{ divides } \mathrm{lt}_\preccurlyeq(g)\}$
- has $t = \#\hat{\mathcal{G}}$ dense rows and $D - t$ identity rows

shape position and lexicographic basis

[Becker-Mora-Marinari-Traverso 1994]

shape position: $\mathcal{G}_{\text{lex}} = \{x_1 - g_1(x_n), \ldots, x_{n-1} - g_{n-1}(x_n), h(x_n)\}$
with $g_1, \ldots, g_{n-1}, h$ univariate in $\mathbb{K}[x_n]$
and $\deg(g_i) < \deg(h) = D$

x_n = smallest variable

g_i = parametrizations

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for **polynomial system solving**:

- solutions $= (g_1(\alpha), \ldots, g_{n-1}(\alpha), \alpha)$ for all roots α of $h(x_n)$
- ensured by generic change of coordinates, if ideal is radical

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computation from the multiplication matrix M_n

$h \in \mathcal{I} \quad \Rightarrow \quad h(x_n)$ is zero in $\mathbb{K}[x]/\mathcal{I}$

- h gives a $\mathbb{K}$-linear combination between $\varepsilon_1, \varepsilon_1 M_n, \ldots, \varepsilon_1 M_n^D$
- the matrix $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_1 M_n \\ \vdots \\ \varepsilon_1 M_n^{D-1} \end{bmatrix} \in \mathbb{K}^{D \times D}$ is invertible (taking $\varepsilon_1 = 1$)

$\Rightarrow$ $h(x_n)$ is the minpoly/charpoly of M_n

previously: dense or sparse linear algebra

using dense linear algebra

$$\begin{bmatrix} -\text{coeffs}(h) & 1 & & \\ -\text{coeffs}(g_1) & & 1 & \\ \vdots & & & \ddots \\ -\text{coeffs}(g_{n-1}) & & & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_1 M_n \\ \vdots \\ \varepsilon_1 M_n^{D-1} \\ \varepsilon_1 M_n^{D} \\ \varepsilon_{x_1} \\ \vdots \\ \varepsilon_{x_{n-1}} \end{bmatrix} = 0$$

- compute matrix in $O(D^\omega \log(D))$

via matrix-matrix products [Keller-Gehrig 1985]

- nullspace computation in $O(D^\omega)$

[Ibarra-Hui-Moran 1982]

deterministic algo in $O(D^\omega \log(D))$

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deterministic algo in $O(D^\omega \log(D))$

using sparse linear algebra [Wiedemann 1986]

for random column vector $r \in \mathbb{K}^{D\times 1}$,
scalar sequence $(\varepsilon_1 M_n^k r)_{0\leqslant k<2D}$
$\rightsquigarrow$ its minimal generator is $h(x_n)$

- compute recurrent sequence in $O(tD^2)$

via matrix-vector products

- find generator h in $\tilde{O}(D)$ [GCD/Padé]
- find $g_1, \ldots, g_{n-1}$ in $\tilde{O}(nD)$ via $n-1$

Hankel systems [Faugère-Mou 2011, 2017]

randomized algo in $O(tD^2)$

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new ingredients

- paradigm shift: ~~sparse~~ $\rightarrow$ structured
- a univariate module with known basis
- from Hermite normal form to lex basis

conclusion | perspectives

paradigm shift: ~~sparse~~ $\rightarrow$ structured

multiplication by x_n in $\mathbb{K}[x_n]/\langle h(x_n)\rangle$

$$\updownarrow$$

companion matrix $M = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -h_0 & -h_1 & \cdots & -h_{D-1} \end{bmatrix}$

with charpoly$(M) = h$

paradigm shift: ~~sparse~~ $\rightarrow$ structured

ideal $\mathcal{I} \subset \mathbb{F}_{29}[x_1, x_2, x_3]$ generated by the $\preccurlyeq_{\text{drl}}$-GB

$$\begin{cases} \mathbf{x_3^4} + 3x_3^3 + 15x_1x_3 + 23x_2x_3 + 3x_3^2 + 26x_2 + 22x_3, \\ \mathbf{x_2x_3^2} + 5x_1x_3 + 28x_2x_3 + 3x_3^2 + 19x_1 + 15x_2 + 17, \\ \mathbf{x_1x_3^2} + 18x_3^3 + 24x_1x_3 + 27x_2x_3 + 19x_3^2 + 2x_1 + 9x_3 + 3, \\ x_2^2 + 12x_1x_3 + 26x_2x_3 + 5x_3^2 + 9x_1 + 6x_2 + 8x_3 + 6, \\ x_1x_2 + 6x_1x_3 + x_2x_3 + 17x_3^2 + 28x_1 + 12x_2 + 8x_3 + 11, \\ x_1^2 + x_1x_3 + 10x_2x_3 + 2x_3^2 + 3x_1 + 16x_2 + 21 \end{cases}$$

- $t = 3$ polynomials with $\preccurlyeq_{\text{drl}}$-leading term divisible by x_3

  the first 3, with leading terms $\mathbf{x_3^4}, \mathbf{x_2x_3^2}, \mathbf{x_1x_3^2}$

- x_3-stability holds

  easily verified: for $\mu \in \{x_2x_3^2, x_1x_3^2, x_3^4\}$, $\frac{x_1}{x_3}\mu$ and $\frac{x_2}{x_3}\mu$ are in $\text{lt}_{\preccurlyeq_{\text{drl}}}(\mathcal{I})$

- zero-dimensional with $D = 8$

  $\preccurlyeq_{\text{drl}}$-monomial basis $\mathcal{B} = (1, x_3, x_3^2, x_3^3, x_2, x_2x_3, x_1, x_1x_3)$

paradigm shift: ~~sparse~~ $\rightarrow$ structured

ideal $\mathcal{I} \subset \mathbb{F}_{29}[x_1, x_2, x_3]$ generated by the $\preccurlyeq_{\mathsf{drl}}$-GB

$$\begin{cases} \mathbf{x_3^4} + 3x_3^3 + 15x_1x_3 + 23x_2x_3 + 3x_3^2 + 26x_2 + 22x_3, \\ \mathbf{x_2x_3^2} + 5x_1x_3 + 28x_2x_3 + 3x_3^2 + 19x_1 + 15x_2 + 17, \\ \mathbf{x_1x_3^2} + 18x_3^3 + 24x_1x_3 + 27x_2x_3 + 19x_3^2 + 2x_1 + 9x_3 + 3, \\ \dots \end{cases}$$

- $t = 3$, $D = 8$
- x_3-stable
- monomial basis $\mathcal{B} = (1, x_3, x_3^2, x_3^3, x_2, x_2x_3, x_1, x_1x_3)$

paradigm shift: ~~sparse~~ $\rightarrow$ structured

ideal $\mathcal{I} \subset \mathbb{F}_{29}[x_1, x_2, x_3]$ generated by the $\preccurlyeq_{\text{drl}}$-GB

$$\begin{cases} x_3^4 + 3x_3^3 + 15x_1x_3 + 23x_2x_3 + 3x_3^2 + 26x_2 + 22x_3, \\ x_2x_3^2 + 5x_1x_3 + 28x_2x_3 + 3x_3^2 + 19x_1 + 15x_2 + 17, \\ x_1x_3^2 + 18x_3^3 + 24x_1x_3 + 27x_2x_3 + 19x_3^2 + 2x_1 + 9x_3 + 3, \\ \dots \end{cases}$$

- $t = 3$, $D = 8$
- x_3-stable
- monomial basis $\mathcal{B} = (1, x_3, x_3^2, x_3^3, x_2, x_2x_3, x_1, x_1x_3)$

multiplication by x_3 in $\mathbb{K}[x_1, x_2, x_3]/\mathcal{I} \longleftrightarrow \mathbb{K}[x_3]$-module structure

$$M = \left[\begin{array}{cccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -9 & -19 & -18 & 0 & -27 & -2 & -24 \end{array}\right] \in \mathbb{K}^{D \times D}$$

paradigm shift: ~~sparse~~ $\rightarrow$ structured

ideal $\mathcal{I} \subset \mathbb{F}_{29}[x_1, x_2, x_3]$ generated by the $\preccurlyeq_{\text{drl}}$-GB

$$\begin{cases} x_3^4 + 3x_3^3 + 15x_1x_3 + 23x_2x_3 + 3x_3^2 + 26x_2 + 22x_3, \\ x_2x_3^2 + 5x_1x_3 + 28x_2x_3 + 3x_3^2 + 19x_1 + 15x_2 + 17, \\ x_1x_3^2 + 18x_3^3 + 24x_1x_3 + 27x_2x_3 + 19x_3^2 + 2x_1 + 9x_3 + 3, \\ \dots \end{cases}$$

- $t = 3$, $D = 8$
- x_3-stable
- monomial basis $\mathcal{B} = (1, x_3, x_3^2, x_3^3, x_2, x_2x_3, x_1, x_1x_3)$

$\preccurlyeq$-Gröbner basis + x_3-stability $\Rightarrow$ basis of $\mathbb{K}[x_3]$-submodule of $\mathcal{I}$

$$M = \left[\begin{array}{cccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -9 & -19 & -18 & 0 & -27 & -2 & -24 \end{array}\right] \in \mathbb{K}^{D \times D}$$

basis of $\mathbb{K}[x_3]$-module $\mathcal{I} \cap (\mathbb{K}[x_3] + x_2\mathbb{K}[x_3] + x_1\mathbb{K}[x_3])$

$$P = \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

paradigm shift: ~~sparse~~ $\rightarrow$ structured

ideal $\mathcal{I} \subset \mathbb{F}_{29}[x_1, x_2, x_3]$ generated by the $\preccurlyeq_{\text{drl}}$-GB

$$\begin{cases} x_3^4 + 3x_3^3 + 15x_1x_3 + 23x_2x_3 + 3x_3^2 + 26x_2 + 22x_3, \\ x_2x_3^2 + 5x_1x_3 + 28x_2x_3 + 3x_3^2 + 19x_1 + 15x_2 + 17, \\ x_1x_3^2 + 18x_3^3 + 24x_1x_3 + 27x_2x_3 + 19x_3^2 + 2x_1 + 9x_3 + 3, \\ \dots \end{cases}$$

- $t = 3$, $D = 8$
- x_3-stable
- monomial basis $\mathcal{B} = (1, x_3, x_3^2, x_3^3, x_2, x_2x_3, x_1, x_1x_3)$

$\preccurlyeq$-Gröbner basis + x_3-stability $\Rightarrow$ basis of $\mathbb{K}[x_3]$-submodule of $\mathcal{I}$

- $\det(P) = \text{charpoly}(M)$
- $\text{Smith}(P) \simeq \text{Frob}(M)$
- [Storjohann 2000]

[Pernet-Storjohann 2007]

- column degrees $(4, 2, 2)$

$$M = \left[\begin{array}{cccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -9 & -19 & -18 & 0 & -27 & -2 & -24 \end{array}\right] \in \mathbb{K}^{D \times D}$$

basis of $\mathbb{K}[x_3]$-module $\mathcal{I} \cap (\mathbb{K}[x_3] + x_2\mathbb{K}[x_3] + x_1\mathbb{K}[x_3])$

$$P = \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

from Hermite normal form to lex basis

$$\begin{matrix} & & \mu_1 = 1 & & \mu_2 = x_2 & & \mu_3 = x_1 \\ & & \vdots & & \vdots & & \vdots \end{matrix}$$

$$\hat{\mathcal{G}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \simeq \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

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Hermite normal form

complexity $O\tilde{}(t^{\omega-1}D)$

[Giorgi-Jeannerod-Villard 2003]
[Gupta-Storjohann 2011]
[Labahn-Neiger-Zhou 2017]

$$\begin{bmatrix} x_3^8 + 26x_3^7 + 8x_3^6 + 17x_3^5 + 19x_3^4 + x_3^3 + 28x_3^2 + 20x_3 + 18 & 0 & 0 \\ 28x_3^7 + 23x_3^6 + 17x_3^5 + 25x_3^4 + 24x_3^3 + 17x_3^2 + 14x_3 + 4 & 1 & 0 \\ 6x_3^7 + 13x_3^6 + 22x_3^5 + 12x_3^4 + 28x_3^3 + 24x_3^2 + 26x_3 + 14 & 0 & 1 \end{bmatrix}$$

from Hermite normal form to lex basis

$$\begin{matrix} & \mu_1 = 1 & & \mu_2 = x_2 & & \mu_3 = x_1 \\ & \vdots & & \vdots & & \vdots \end{matrix}$$

$$\hat{\mathcal{G}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \simeq \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

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row $i \simeq$ polynomial $p_{i1}(x_3) + p_{i2}(x_3)x_2 + p_{i3}(x_3)x_1$

$$x_3^8 + 26x_3^7 + 8x_3^6 + 17x_3^5 + 19x_3^4 + x_3^3 + 28x_3^2 + 20x_3 + 18,$$
$$x_2 + 28x_3^7 + 23x_3^6 + 17x_3^5 + 25x_3^4 + 24x_3^3 + 17x_3^2 + 14x_3 + 4,$$
$$x_1 + 6x_3^7 + 13x_3^6 + 22x_3^5 + 12x_3^4 + 28x_3^3 + 24x_3^2 + 26x_3 + 14$$

from Hermite normal form to lex basis

$$\mu_1 = 1 \qquad \mu_2 = x_2 \qquad \mu_3 = x_1$$

$$\hat{\mathcal{G}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \simeq \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

Hermite normal form

complexity $O\tilde{}(t^{\omega-1}D)$

[Giorgi-Jeannerod-Villard 2003]
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$$\begin{bmatrix} x_3^8 + 26x_3^7 + 8x_3^6 + 17x_3^5 + 19x_3^4 + x_3^3 + 28x_3^2 + 20x_3 + 18 & 0 & 0 \\ 28x_3^7 + 23x_3^6 + 17x_3^5 + 25x_3^4 + 24x_3^3 + 17x_3^2 + 14x_3 + 4 & 1 & 0 \\ 6x_3^7 + 13x_3^6 + 22x_3^5 + 12x_3^4 + 28x_3^3 + 24x_3^2 + 26x_3 + 14 & 0 & 1 \end{bmatrix}$$

row $i \simeq$ polynomial $p_{i1}(x_3) + p_{i2}(x_3)x_2 + p_{i3}(x_3)x_1$

$$x_3^8 + 26x_3^7 + 8x_3^6 + 17x_3^5 + 19x_3^4 + x_3^3 + 28x_3^2 + 20x_3 + 18,$$
$$x_2 + 28x_3^7 + 23x_3^6 + 17x_3^5 + 25x_3^4 + 24x_3^3 + 17x_3^2 + 14x_3 + 4,$$
$$x_1 + 6x_3^7 + 13x_3^6 + 22x_3^5 + 12x_3^4 + 28x_3^3 + 24x_3^2 + 26x_3 + 14$$

= lex basis

outline

problem and result

- problem: change of monomial order
- faster algorithm via univariate matrices
- improved software performances

previous work

- stability and multiplication matrix
- shape position and lexicographic basis
- previously: dense or sparse linear algebra

new ingredients

- paradigm shift: ~~sparse~~ $\rightarrow$ structured
- a univariate module with known basis
- from Hermite normal form to lex basis

conclusion | perspectives

- improved complexity bound and faster software implementation
- based on the identification and exploitation of an algebraic structure
  $\rightsquigarrow$ $\mathbb{K}[x_n]$-modules and univariate polynomial matrix computations
- relating the Hermite normal form of a $\mathbb{K}[x_n]$-submodule of $\mathcal{I}$ and the lexicographic Gröbner basis of the ideal $\mathcal{I}$

summary

conclusion

perspectives

- software: add SIMD vectorization + integrate into msolve
  https://github.com/algebraic-solving/msolve
  https://msolve.lip6.fr/
- handle case with $\mathcal{I}$ non-radical but $\sqrt{\mathcal{I}}$ in shape position?
- relax assumptions about stability and shape position?