Efficient Algorithms in Computer Algebra (2024–2025)

https://wikimpri.dptinfo.ens-cachan.fr/doku.php?id=cours:c-2-22

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Introduction — Computer Algebra and Complexity — Fast Multiplication

Part I

Introduction

Organisation

Practical questions

- language: English or French?
- course is “breakable”
  - following the second part only is not reasonable
  - following the first part only is feasible (but not recommended)

Professors

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Organisation

Material

- webpage for the course, with info and material (frequent updates):
  https://wikimpri.dptinfo.ens-cachan.fr/doku.php?id=cours:c-2-22
- all slides in English
- book in French, printed 2017 version is cheap ($\approx$ 15€)
- updated pdf, legally cost-free, is available here:
  https://hal.archives-ouvertes.fr/AECF/

Calendar

- always refer to the webpage! ask us by email in case of doubts
- **time:** Mondays, 16:15-19:15; **location:** room 1004
- first period, **exception: Thursday 14/11**

Organisation

How to work?

- with pen and paper + with a computer (SageMath, Maple, ...)
- in class: pay close attention, be proactive, ask questions
- at home: weekly regular work $\gg$ intense sprint 4 days before exam

To help you towards this:

- basic questions/exercises/examples/demos during each class
  take advantage of them!
- exercises proposed at the end of each session
  - it is in your interest to study them during the week
  - beginning of next session: one of you volunteers to correct it
- 3-hour tutored exercise session on 18/11
  practice on exam-like exercises, with at least one professor available for you

Master's Internships?

Interested? Let's discuss soon!

- we provide research internship subjects in our respective teams
  MathExp, Inria Saclay & MAX, Polytechnique & PolSys, LIP6 / Sorbonne U.
- we can also provide advice for other internships/PhD opportunities with colleagues in computer algebra-related domains
  in Bordeaux, Grenoble, Lille, Limoges, Lyon, Montpellier, Nancy, Rennes, Toulouse, …
- wide range of topics, with common denominators:
  - involve a variable, but nonnegligible, amount of mathematics/algebra
  - questions of effectiveness/efficiency of computations

Master's Internships?

Possible research subjects:

- supervised by MAX team @ LIX/Polytechnique (contact: Mezzarobba)
  - Numeric-symbolic polynomial system solving
    https://dl.vwx.fr/PoK06ZjlaYw1uFst/stage-odelix-polsys.pdf
  - Numerical approach of generalized flatness
    https://www.lix.polytechnique.fr/max/node/stage-node-e.en.html
- Algorithms for algebraic approximants and guess-and-prove approaches
  supervision/contact: Bostan & Neiger; detailed outline available soon
- examples of other subjects recently proposed in our teams:
  - Fast evaluation of elementary functions with medium precision
  - Sparse interpolation of rational functions
  - Computing contiguity/multiplication matrices for statistical physics
  - Algebraic cryptanalysis of new NIST multivariate signature schemes
  - Méthodes algébriques pour le calcul de topologies d'ensembles semi-algébriques

contact us asap if interested

Master's Internships?

ISSAC 2024 conference topics: Algorithmic aspects

- Exact and symbolic linear, polynomial and differential algebra
- Symbolic-numeric, homotopy, perturbation and series methods
- Computational algebraic geometry, group theory and number theory, quantifier elimination and logic
- Computer arithmetic
- Summation, recurrence equations, integration, solution of ODEs & PDEs
- Symbolic methods in other areas of pure and applied mathematics
- Complexity of algebraic algorithms and algebraic complexity

ISSAC 2024 conference topics: Software aspects

- Design of symbolic computation packages and systems
- Language design and type systems for symbolic computation
- Data representation
- Considerations for modern hardware
- Algorithm implementation and performance tuning
- Mathematical user interfaces
- Use with systems for, e.g., digital libraries, courseware, simulation and optimization, automated theorem-proving, computer-aided design, and automatic differentiation.

Contents

This year:

- power series, polynomials, matrices
- linear recurrences and linear differential equations
- polynomial matrices and Hermite-Padé approximation
- factorization of polynomials, lattice reduction
- binomial sums
- opening to combinatorics

Related topics / courses:

- techniques in cryptography and cryptanalysis $\to$ C-2-12-1
- arithmetic algorithms for cryptology $\to$ C-2-12-2
- error correcting codes and applications to cryptography $\to$ C-2-13-2
- analysis of algorithms $\to$ C-2-15

Our viewpoint

Computer Algebra = design of fast algebraic algorithms on exact representations of mathematical objects in the computer.

This is a part of “doing mathematics” on the computer.

Motivating Examples

A computational proof of

Theorem (Ramanujan)

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}$$

is by combining resultants for elimination.

Motivating Examples

A computational proof of:

Theorem (Apéry, 1978)

$\zeta(3) := 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \notin \mathbb{Q}.$ "$\zeta(3)$ is irrational."

Proof: relies crucially on proving that both

$$a_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

$$\text{and} \qquad b_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$n^3 a_n - (2n-1)(17n^2 - 17n + 5)a_{n-1} + (n-1)^3 a_{n-2} = 0 \qquad (n \geq 2).$$

Recurrence can be discovered and verified in a few seconds!

On the computer

List of Computer Algebra Systems (subjective selection)

Axiom, CoCoA, Derive, GAP, Macaulay2, Magma, Maple, Mathemagix, Mathematica, Maxima, MuPAD, PARI/GP, Reduce, SageMath, Singular

- more recently: development of optimized open-source libraries
- SageMath gathers many such state-of-the-art libraries

what you can compute in about 1 second

with fflas-ffpack			with NTL		
`> PLUQ`	$m = 3800$	1.00s	`> PolMul`	$d = 7 \times 10^6$	1.03s
`> LinSys`	$m = 3800$	1.00s	`> Division`	$d = 4 \times 10^6$	0.96s
`> MatMul`	$m = 3000$	0.97s	`> XGCD`	$d = 2 \times 10^5$	0.99s
`> Inverse`	$m = 2800$	1.01s	`> MinPoly`	$d = 2 \times 10^5$	1.10s
`> CharPoly`	$m = 2000$	1.09s	`> MPeval`	$d = 1 \times 10^4$	1.01s

Part II

Computer algebra and computability/effectiveness

A negative result

The Richardson–Matiyasevich Theorem

In the class of expressions obtained from a variable X and the constant 1 by application of the ring operations +, $-$, $\times$ and composition with the function $\sin(\cdot)$ and the absolute value function $|\cdot|$, the test of equivalence to 0 is undecidable.

- equality test is a zero test (as soon as subtraction exists)
- no "good" `simplify()`; it is made of heuristics
- computer algebra: work with algebraic constructs that preserve the decidability of the zero test

Effective algebraic structures

Definition

An algebraic structure (group, ring, field, vector space, …) is *effective* if it is endowed with:

- a data structure to represent its elements;
- algorithms to perform its inner operations and to test equality and other predicates.

- e.g. an effective ring comes with algorithms for: equality, addition, subtraction, multiplication
- composition of data structures via lists/arrays

Integers and variants

Machine integers / word-size integers

- processor "integers" = integers modulo 2^w (usually $w = 32$ or $w = 64$)
- operations $=, +, -, \times$ in hardware

Big integers, a.k.a. *bignums*

- unique writing $N = (-1)^\varepsilon \times (a_0 + a_1 B + \cdots + a_k B^k)$ for a fixed base B
- Lemma: the ring $\mathbb{Z}$ of relative integers is effective.

Modular integers

- core tool: Euclidean division with remainder in $\mathbb{Z}$
- Lemma: for any integer $n \geq 2$, the ring $\mathbb{Z}/n\mathbb{Z}$ is effective

$\rightsquigarrow$ avoid intermediate expression swell

$\rightsquigarrow$ reconstructions by the Chinese Remainder Theorem (CRT)

$\rightsquigarrow$ allows probabilistic heuristics by calculations modulo n

$\rightsquigarrow$ algorithms with deterministic outputs by controlling sizes and bad n

Vectors and matrices

Vector: typically an array of pointers to the coefficients
(or simply, a memory-contiguous array if a coefficient fits into a machine word)

Proposition

If $\mathbb{K}$ is an effective field,

- the vector space $\mathbb{K}^n$ is effective,
- the ring $\mathcal{M}_n(\mathbb{K}) = \mathbb{K}^{n\times n}$ is effective.

Depending on the application, dense or sparse representation
$\rightarrow$ different algorithms!

Polynomials, fractions

Depending on the application, dense or sparse representation
$\rightarrow$ different algorithms!

Proposition

If $\mathbb{A}$ is an effective ring, then so is $\mathbb{A}[X]$.

- multivariate polynomials by iteration
  (not necessarily done this way in practice)
- strong connection between univariate polynomials and big integers

Proposition

If $\mathbb{A}$ is an effective domain, then its fraction field is effective.

- provides $\mathbb{Q}$ and $\mathbb{K}(X)$
- representation variants: $\mathrm{Frac}(\mathbb{Z}[X, Y])$, $\mathrm{Frac}(\mathrm{Frac}(\mathrm{Frac}(\mathbb{Z})[X])[Y])$, …

Truncated formal power series

- truncated series $a_0 + a_1X + \cdots + a_{N-1}X^{N-1} + O(X^N)$
  $\rightsquigarrow$ represented as a polynomial of degree $< N$
- optimized algorithm for the *short product*
  [Schönhage: "*Never waste a factor of 2!*"]

Proposition

If $\mathbb{A}$ is an effective ring and if $N \in \mathbb{N}$, then $\mathbb{A}[X]/(X^N)$ is an effective ring.

- computing approximations
- representing (exactly) rational fractions if numerators and denominators are with bounded degrees
- reconstructing linear differential equations with polynomial coefficients (*guessing*); analogue for linear recurrence equations
- reconstructing bivariate polynomials from univariate series solutions: for factorization and for solving polynomial systems

Equations as the right data structures

Non-explicit or infinite mathematical objects can be represented exactly on the computer when they are solutions to finite equations:

- $\sqrt{2}$ = just a symbol whose square is 2,
- $\ln x$ = just a symbol whose derivative is $1/x$.

$\longrightarrow$ algorithms on **implicit representations**

Algebraic numbers = univariate polynomials

Proposition

If $\mathbb{K}$ is an effective field, then so is its algebraic closure $\bar{\mathbb{K}}$.

Calculations by resultants, series, gcd.

Consequence: “easy” computational proof of

$$\frac{\sin(\frac{2\pi}{7})}{\sin^2(\frac{3\pi}{7})} - \frac{\sin(\frac{\pi}{7})}{\sin^2(\frac{2\pi}{7})} + \frac{\sin(\frac{3\pi}{7})}{\sin^2(\frac{\pi}{7})} = 2\sqrt{7}$$

⌨ SageMath example </>

Polynomial systems

Proposition

Let $\mathbb{K}$ be an effective field, and $f_1, \ldots, f_m$ be polynomials from the ring $\mathbb{K}[X_1, \ldots, X_n]$. Then the quotient ring $\mathbb{K}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is effective.

- algorithms by resultants or by Gröbner bases
- very strong connection to geometry

Linear differential equations, linear recurrence equations

If $\mathbb{K}$ is an effective field, the set…

$$\left\{\text{formal power series } \sum_{n\in\mathbb{N}} a_n X^n \in \mathbb{K}[[X]] \text{ that are solutions to} \right.$$
$$\left. \text{linear differential equations with coefficients from } \mathbb{K}[X]\right\}$$

is an effective ring.

If $\mathbb{K}$ is an effective field, the set…

$$\left\{\text{sequences } (a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \text{ that are solutions to} \right.$$
$$\left. \text{linear recurrences with coefficients in } \mathbb{K}[n]\right\}$$

is an effective ring.

- special functions in mathematical physics; combinatorial sequences
- algorithms by a non-commutative variant of resultants
- equality test reduces to the identification of initial conditions

Systems of linear differential and recurrence equations

Algorithms (of *creative telescoping*) make possible the automatic proof of identities like:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = n2^{3n-1} + 2^{3n} - 3n2^{n-2}\binom{2n}{n},$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}.$$

(binomial coefficients, Bessel functions, etc)

Part III

Computer algebra and complexity/efficiency

Measures of (time) complexity

Random access machines (*RAM*)

- two tapes + arbitrarily many registers, all containing integers
- read, write, addition, subtraction, product, division, jumps

Arithmetic complexity

- counts operations on some effective algebraic structure $\mathbb{A}$ (arithmetic operations and tests of predicates)
- does not count copies, operations on loop counters, indirections

- okay if operations in $\mathbb{A}$ are preponderant and on similar sizes
- caution: memory is neglected, e.g. matrix transposition is free, etc.

Bit complexity

- counts operations on digits of integers written in B
- better if intermediate calculations are with integers of variable sizes

Notation $O(\cdot)$ and variations

Various "size" parameters of data structures: number of digits, degrees, matrix dimensions, etc.

Comparaisons of complexities as functions of "sizes"

- $f(n) = O(g(n))$ as $n \to \infty$ if $\exists K > 0, \exists N > 0, \forall n > N,\ |f(n)| \leq K\,|g(n)|$
- be cautious if there are several parameters
- simplification to hide logarithms: $f(n) = \tilde{O}(g(n))$ as $n \to \infty$ if $\exists k \geq 0, f(n) = O(g(n) \log^k |g(n)|)$
- $f(n) = \Theta(g(n))$ as $n \to \infty$ if $\exists K > 0, \exists K' > 0, \exists N > 0, \forall n > N,\ K'\,|g(n)| \leq |f(n)| \leq K\,|g(n)|$
- remark: $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $g(n) = O(f(n))$

Caution: a few chapters in the book write $f(X) := g(X) + O(X^N)$ to mean "compute and store the polynomial remainder $f(X) := \mathrm{rem}(g(X), X^N)$". This is a ternary notation "? := ? + $O(X^?)$", not to be confused with "? = ? + $O(X^?)$".

What efficiency?

- generally speaking: worst-case time complexity
- often, this reflects the average-case complexity
  at least for operations concerning fundamental algebraic structures

An algorithm is quasi-optimal...

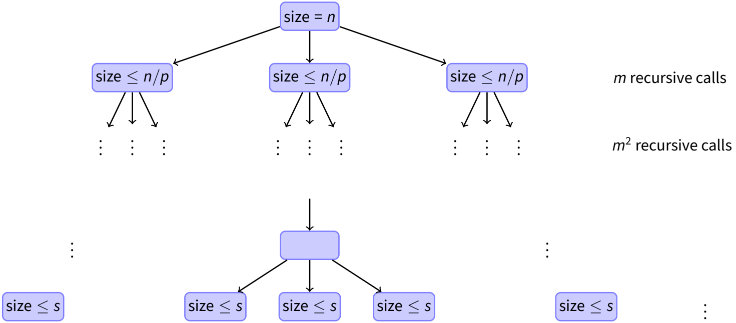
if its complexity is some $\tilde{O}$ of the sum of the sizes of its input and output

(We will slightly extend this definition for linear algebra.)

ultimate goal

design quasi-optimal algorithms

The programming paradigm “divide and conquer” (DAC)



m recursive calls, on p times smaller data, down to threshold $s \geq p$

When is this of interest? Where are the most costly manipulations?

Complexity analysis of “divide and conquer”

Theorem

Let T be an increasing function and let C be a function ruled by the inequality

$$C(n) \leq \begin{cases} T(n) + mC(\lceil n/p \rceil), & \text{if } n \geq s \geq p \\ \kappa & \text{otherwise,} \end{cases}$$

with $m > 0$ and $\kappa > 0$, and so that there exist q and r with $1 < q \leq r$ satisfying

$$qT(n) \leq T(pn) \leq rT(n), \qquad \text{for all sufficiently large } n.$$

Then, when $n \to \infty$, *[dominant cost:]*

$$C(n) = \begin{cases} O(T(n)), & \text{if } q > m, & \textit{[top of tree]} \\ O(T(n)\log_p n), & \text{if } q = m, & \textit{[all levels]} \\ O\left(n^{\log_p(m/p)}T(n)\right) & \text{if } q < m. & \textit{[bottom of tree]} \end{cases}$$

Part IV

Fast polynomial multiplication

A long history (and a spoiler)

Theorem (for the dense representation and coefficients in a ring $\mathbb{A}$)

The multiplication of polynomials of degree at most n in $\mathbb{A}[X]$ requires:

- $O(n^2)$ operations in $\mathbb{A}$ by the naive algorithm (dates back to Antiquity);
- $O(n^{\log_2 3})$ ops in $\mathbb{A}$ by the algorithm by Karatsuba (and Ofman) (1963);
- $O(n \log n)$ operations in $\mathbb{A}$ when $\mathbb{A}$ contains enough "good" roots of unity
  via Fast Fourier Transform (FFT), known to Gauss (1805), rediscovered by Cooley and Tukey (1965)
- $O(n \log n \log \log n)$ operations in $\mathbb{A}$ by the algorithm by Schönhage and Strassen (1971)
  generalizing the FFT applicability by introducing "virtual" roots of unity
- $O(n \log n \log \log n)$ operations in $\mathbb{A}$ by Cantor and Kaltofen (1991), for general $\mathbb{A}$
  arbitrary (possibly non-commutative) algebra, $O(n \log n)$ mul. and $O(n \log n \log \log n)$ add./sub.

recently, after the breakthrough of Fürer's algorithm (2007) which multiplies integers of size n in $O(n \log n\, K^{\log^* n})$ bit operations (constant K):

- *bit complexity* $O(n \log(p) \log(n \log(p))\, 8^{\log^*(n \log(p))})$ by Harvey, van der Hoeven, Lecerf (JACM 2017), for polynomials over $\mathbb{F}_p$ with p prime
  $\log^* n$ = minimal number k such that $\log^{\circ k} n \leq 1$
- under a number-theoretic conjecture: for polynomials over $\mathbb{F}_q$, *bit complexity* $O(n \log(q) \log(n \log(q)))$ (Harvey and van der Hoeven, JACM 2022).

Naive multiplication algorithm

$$F = f_0 + \cdots + f_n X^n,\ G = g_0 + \cdots + g_n X^n \quad \longrightarrow \quad H := FG = h_0 + \cdots + h_{2n} X^{2n}$$

$$\boxed{h_i = \sum_{j=0}^{i} f_j g_{i-j}, \qquad h_{2n-i} = \sum_{j=0}^{i} f_{n-j} g_{n-i+j}, \qquad (0 \le i \le n)}$$

$$\sum_{i=0}^{n} \begin{pmatrix} i+1 \text{ multiplications} \\ i \text{ additions} \end{pmatrix} + \sum_{i=0}^{n-1} \begin{pmatrix} i+1 \text{ multiplications} \\ i \text{ additions} \end{pmatrix} = \begin{pmatrix} (n+1)^2 \text{ multiplications} \\ n^2 \text{ additions} \end{pmatrix}$$

Karatsuba's multiplication: the idea in degree 1

$$F = f_0 + f_1 X,\ G = g_0 + g_1 X \quad \longrightarrow \quad H := FG = h_0 + h_1 X + h_2 X^2$$

Naively, 4 multiplications: $h_0 = f_0 g_0,\ h_1 = f_0 g_1 + f_1 g_0,\ h_2 = f_1 g_1$

Some easy evaluations

(up to some interpretation at infinity)

$$h_0 = H(0) = F(0)G(0) = f_0 g_0$$

$$h_0 + h_1 + h_2 = H(1) = F(1)G(1) = (f_0 + f_1)(g_0 + g_1)$$

$$h_2 = H(\infty) = F(\infty)G(\infty) = f_1 g_1$$

Gain of one multiplication

$$\boxed{h_1 = (f_0 + f_1)(g_0 + g_1) - f_0 g_0 - f_1 g_1}$$

Karatsuba's algorithm

Input F, G of degrees at most $n-1$.

Output $H = FG$.

1. If $n = 1$, return FG.
2. Set $k = \lceil n/2 \rceil$ and decompose F et G according to the equation

$$F = F^{(0)} + F^{(1)}X^k, \qquad G = G^{(0)} + G^{(1)}X^k,$$

3. Recursively compute $H_0 = F^{(0)}G^{(0)}$ and $H_2 = F^{(1)}G^{(1)}$.
4. Compute $A = F^{(0)} + F^{(1)}$ et $B = G^{(0)} + G^{(1)}$.
5. Recursively compute $C = AB$.
6. Compute $H_1 = C - H_0 - H_2$.
7. Return $H_0 + H_1X^k + H_2X^{2k}$.

Remark: A, B have degree $< k$; H_0, H_1, H_2, C have degree $< n$.

Complexity analysis of Karatsuba's algorithm

Theorem

If n is a power of 2, Karatsuba's algorithm computes the product of two polynomials of degrees at most $n-1$ in at most $9n^{\log_2 3}$ operations in $\mathbb{A}$.

Proof: For $n = 2^\ell$, that is to say $\ell = \log_2 n$:

$$\begin{aligned} K(n) &\leq 3K(n/2) + 4n \\ &\leq 3^2 K(n/2^2) + 4n(3/2 + 1) \\ &\leq \dots \\ &\leq 3^\ell K(n/2^\ell) + 4n((3/2)^{\ell-1} + \cdots + 3/2 + 1) \\ &\leq 3^\ell K(1) + 4n \frac{(3/2)^\ell - 1}{3/2 - 1} \\ &\leq 3^\ell (1 + 4 \cdot 2) = 9n^{\log_2 3}. \end{aligned}$$

Algorithms by Cook (1963), Toom (1966), Schönhage (1966), Knuth (1969)

Theorem

For a given $\epsilon > 0$, by increasing the number of evaluation points one obtains an algorithm of complexity $O(n^{1+\epsilon})$.

Multiplication by DFT (*Discrete Fourier Transform*): the idea

Relies on a suitable choice of points for evaluation/interpolation

$$\begin{aligned} \mathrm{DFT} : \mathbb{A}[X]_{<n} &\xrightarrow{\sim} \mathbb{A}^n \\ P &\longmapsto \big(P(\omega^0), \dots, P(\omega^{n-1})\big) \end{aligned}$$

Input F and G two polynomials, n an integer, and ω a principal nth root of unity *(definition to come)*.

Output $\mathrm{rem}(FG, X^n - 1)$.

1. *Precomputation.* Compute the powers $\omega^2, \dots, \omega^{n-1}$.
2. *Evaluation.* Compute $(u_i)_{i=0}^{n-1} := \mathrm{DFT}(F)$ et $(v_i)_{i=0}^{n-1} := \mathrm{DFT}(G)$.
3. *Coordinate-wise product.* Compute $W := (u_i v_i)_{i=0}^{n-1}$.
4. *Interpolation.* Compute and return $\mathrm{DFT}^{-1}(W)$.

$\boxed{\text{there remains to design a fast algorithm for DFT and } \mathrm{DFT}^{-1}}$

Roots of unity of a general ring $\mathbb{A}$

Finer and finer definitions

- ω is an nth root of unity if $\omega^n = 1$
- ω is a primitive nth root of unity if $\omega^n = 1$ and if $0 < t < n \Rightarrow \omega^t - 1$ non-zero
- ω is a principal nth root of unity if $\omega^n = 1$ and if $0 < t < n \Rightarrow \omega^t - 1$ non-zero and non-zero-divisor

Properties

- ω is a primitive nth root of $1 \Rightarrow \omega^{-1}$ is a primitive nth root of 1
- $n = pq$ and ω is a primitive nth root of $1 \Rightarrow$
  ω^p is a primitive qth root of 1
- ω is a primitive nth root of 1 and $0 < \ell < n \Rightarrow \sum_{j=0}^{n-1} \omega^{\ell j} = 0$
- three analogous statements for principal roots of unity

DFT (evaluation) by FFT (*Fast Fourier Transform*)

Input $P = p_0 + \cdots + p_{n-1}X^{n-1}$; the powers $1, \omega, \cdots, \omega^{n-1}$ of some principal nth root of unity ω, n being a power of 2.

Output $\big(P(\omega^0), \ldots, P(\omega^{n-1})\big)$.

1. If $n = 1$, return p_0, otherwise, set $k = n/2$ and calculate
$$R_0(X) := \sum_{j=0}^{k-1} (p_j + p_{j+k})X^j, \qquad \bar{R}_1(X) := R_1(\omega X) = \sum_{j=0}^{k-1} (p_j - p_{j+k})\omega^j X^j.$$
2. Recursively compute the DFT of R_0 and $\bar{R}_1$ on the family $\big(1, \omega^2, \ldots, (\omega^2)^{k-1}\big)$. *("time decimation")*
3. Return $\Big(R_0(1), \bar{R}_1(1), R_0(\omega^2), \bar{R}_1(\omega^2), \ldots, R_0\big((\omega^2)^{k-1}\big), \bar{R}_1\big((\omega^2)^{k-1}\big)\Big)$.

Correctness

$$n = 2k \text{ and } \omega \text{ is a primitive/principal } n\text{th root of } 1 \Rightarrow \omega^k = -1$$

$$P = (X^k - 1)Q_0 + R_0 = (X^k + 1)Q_1 + R_1 \Rightarrow P(\omega^\ell) = \begin{cases} R_0(\omega^\ell) & \text{if } \ell \text{ even,} \\ R_1(\omega^\ell) & \text{if } \ell \text{ odd.} \end{cases}$$

Complexity analysis of the FFT algorithm

Theorem

For n a power of 2, Fast Fourier Transform (FFT) requires $\simeq \frac{3n}{2} \log n$ operations in $\mathbb{A}$. Each multiplication in $\mathbb{A}$ done by the algorithm is between an element of $\mathbb{A}$ and some power of ω.

Proof: For $n = 2^\ell$, that is to say $\ell = \log_2 n$:

$$\begin{aligned}
F(n) &\leq 2F(n/2) + \frac{3n}{2} \\
&\leq 2^2 F(n/2^2) + \frac{3n}{2}(2/2 + 1) \\
&\leq \dots \\
&\leq 2^\ell F(n/2^\ell) + \frac{3n}{2}(2^{\ell-1}/2^{\ell-1} + \dots + 2/2 + 1) \\
&\leq nF(1) + \frac{3n}{2}\ell = \frac{1 + 3\log_2 n}{2} n.
\end{aligned}$$

Interpolate is evaluate (!)

Given the Vandermonde matrix $V_\omega := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & & & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)^2} \end{pmatrix}$, we have:

$$\mathrm{DFT}(P) = \big(P(\omega^0), \ldots, P(\omega^{n-1})\big) = (p_0, \ldots, p_{n-1}) V_\omega .$$

Lemma

If $\omega \in \mathbb{A}$ is a principal nth root of unity, then $V_{\omega^{-1}} V_\omega = n I_n$.

Proof:

$$\sum_{k=0}^{n-1} \omega^{-(i-1)k} \omega^{k(j-1)} = \sum_{k=0}^{n-1} \omega^{(j-i)k} = n\delta_{i,j}.$$

Said differently: $(\mathrm{DFT}_\omega)^{-1} = \frac{1}{n} \mathrm{DFT}_{\omega^{-1}}$.

Complexity analysis of multiplication by FFT when $\mathbb{A}$ contains roots of unity for all $n = 2^k$

Input F and G two polynomials, n an integer, and ω a principal nth root of unity, assumed to exist in $\mathbb{A}$.

Output FG, assumed to be of degree $< n$, a power of 2.

1. *Precomputation.* Calculate the powers $\omega^2, \dots, \omega^{n-1}$.
2. *Evaluation.* Compute $(u_i)_{i=0}^{n-1} := \mathrm{DFT}_\omega(F)$ and $(v_i)_{i=0}^{n-1} := \mathrm{DFT}_\omega(G)$ by FFT.
3. *Coordinate-wise product.* Compute $W := (u_i v_i)_{i=0}^{n-1}$.
4. *Interpolation.* Compute, using FFT, and return $\frac{1}{n}\mathrm{DFT}_{\omega^{-1}}(W)$.

Theorem

If 2 is invertible in $\mathbb{A}$, if n is some power of 2, and if ω is a principal nth root of unity in $\mathbb{A}$, the product of two polynomials whose sum of degrees is $< n$ can be computed in $\frac{9}{2} n \log n + O(n)$ operations in $\mathbb{A}$. Only n of the products are between two elements of $\mathbb{A}$ that are general elements (that is, not powers of ω or $1/n$).

"Practical" remark: Fourier primes

Proposition

The finite field $\mathbb{F}_q$ with q elements contains a primitive nth root of unity if and only if n divides $q-1$

Good prime numbers

A prime p is called *FFT prime* if it has the form $p = 2^e\ell + 1$ for e "big enough"

$$p := 4179340454199820289 = 29 \times 2^{57} + 1,\ \mathbb{A} := \mathbb{F}_p,\ n := 2^{57},$$
$\omega := 21$ is a primitive nth root of unity

Sketch of the Schönhage–Strassen algorithm

“Virtual” roots of unity

If 2 is invertible in $\mathbb{A}$ and if n is a power of 2, then $\omega = X + (X^n + 1)$ is a principal $(2n)$th root of 1 in $\mathbb{A}[X]/(X^n + 1)$ (which is not always a domain, even for a field $\mathbb{A}$).

Input F and G of degrees $< n = 2^k$, for $k > 2$.

Output $\operatorname{rem}(FG, X^n + 1)$.

1. Let $d = 2^{\lfloor k/2 \rfloor}$ and $\delta = n/d$. Rewrite F and G in the form
$$\bar{F}(X,Y) = F_0(X) + \cdots + F_{\delta-1}(X)Y^{\delta-1}, \quad \bar{G}(X,Y) = G_0(X) + \cdots + G_{\delta-1}(X)Y^{\delta-1},$$
with F_i, G_i of degrees $< d$ and s.t. $F(X) = \bar{F}(X, X^d)$ and $G(X) = \bar{G}(X, X^d)$.
2. Compute $\bar{H} := \operatorname{rem}(\bar{F}\bar{G}, Y^\delta + 1)$ in $\mathbb{B}[Y]$ by a variation of FFT, where $\mathbb{B} = \mathbb{A}[X]/(X^{2d} + 1)$ and by recursive calls for products in $\mathbb{B}$.
3. Return $H(X, X^d)$.

Theorem

Let $\mathbb{A}$ be a ring in which 2 is invertible, with known inverse. Then, two polynomials of $\mathbb{A}[X]$ of degrees $< n$ can be multiplied in $O(n \log n \log\log n)$ operations in $\mathbb{A}$.

Multiplication functions

Abstraction of cost functions	$\longrightarrow$	Expression of complexity independent of the multiplication algorithm

Definition

$\mathsf{M} : \mathbb{N}_{>0} \to \mathbb{R}_{>0}$ is a *multiplication function for* $\mathbb{A}[X]$ if:

- all P, Q of degree $< n$ in $\mathbb{A}[X]$ can be multiplied in at most $\mathsf{M}(n)$ arithmetic operations in $\mathbb{A}$;
- $n \mapsto \mathsf{M}(n)/n$ is an increasing function of $n \in \mathbb{N}_{>0}$;
- for all m and n of $\mathbb{N}_{>0}$, $\mathsf{M}(mn) \leq m^2 \mathsf{M}(n)$.

Properties

- *(superlinearity)* $n \leq \mathsf{M}(n)$; $\mathsf{M}(m) + \mathsf{M}(n) \leq \mathsf{M}(m+n)$; $m\,\mathsf{M}(n) \leq \mathsf{M}(mn)$.
- *(usual special cases)* $2\mathsf{M}(n) \leq \mathsf{M}(2n)$; $\sum_i \mathsf{M}(n_i) \leq \mathsf{M}\left(\sum_i n_i\right)$.
- *(at most quadratic)* $\mathsf{M}(n) \leq n^2$.