Polynomial matrices
introduction, motivations, basic algorithms

Exercises for 14 December 2023

Exercise 1. Matrix equation $AU = V$. Let $A \in \mathbb{K}[X]^{m \times m}$ be nonsingular with all entries of degree $\leq d_1$, let $V \in \mathbb{K}[X]^{m \times k}$ with all entries of degree $\leq d_2$.

Two typical cases of interest for the equation $AU = V$ are $k = 1$ (linear system solving over $\mathbb{K}[X]$), and $k = m$ with $V = I_m$ (inversion of $A$).

1. Show that $A^{-1}V$ can be represented as a fraction with numerator a matrix $U$ in $\mathbb{K}[X]^{m \times k}$ and denominator a polynomial $\Delta$ in $\mathbb{K}[X]$.
2. Give an upper bound on $\deg \det(A)$.
3. Give upper bounds that you can require on $\deg(\Delta)$ and on the degrees of entries of $U$ (i.e. there exists a couple $(U, \Delta)$ for question 1 which satisfies these bounds).
4. Prove that $A^{-1} \in \mathbb{K}[X]^{m \times m} \iff \det(A) \in \mathbb{K} \setminus \{0\}$.

Remark: matrices with determinant in $\mathbb{K} \setminus \{0\}$ are called unimodular.

Exercise 2. Using the evaluation-interpolation paradigm,

1. Give a multiplication algorithm for matrices in $\mathbb{K}[X]^{m \times m}$.
2. Give a determinant algorithm.
3. Give an inversion algorithm (finding the inverse over the fractions $\mathbb{K}(X)$).

Hints: exploit known degree bounds on the output to determine the number of points to use; for inversion, you can assume that you have at your disposal a quasi-linear algorithm for Cauchy interpolation (see the slides for references).

For each of these algorithms,

1. Give the lower bound it requires on the cardinality of $\mathbb{K}$.
2. State and prove an upper bound on its complexity.

Further perspective: could your complexity bounds take into account degree measures that refine the matrix degree such as the average row degree?