Polynomial matrices approximation and interpolation

Exercises for 2023-12-21

Recall the problem of *Vector rational interpolation* studied in last lecture:

Input:	• vector of polynomials $\mathbf{F} = [f_1 \ \cdots \ f_m]^\mathsf{T} \in \mathbb{K}[X]^{m\times 1}$; • points $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{K}^d$; • shift $\boldsymbol{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m$.
Output:	matrix $\mathbf{P} \in \mathbb{K}[X]^{m\times m}$ such that $\mathbf{P}$ is $\boldsymbol{s}$-reduced and the rows of $\mathbf{P}$ form a basis of the $\mathbb{K}[X]$-module $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$.

Here, the $\mathbb{K}[X]$-submodule of $\mathbb{K}[X]^{1\times m}$ is defined as:

$$\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F}) = \left\{ \mathbf{p} \in \mathbb{K}[X]^{1\times m} \mid \mathbf{pF} = 0 \bmod \prod_{1\leqslant i\leqslant d} (X - \alpha_i) \right\}.$$

Recall that "$\mathbf{P}$ is a basis of $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$" means that each row of $\mathbf{P}$ is in $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$, and that any $\mathbf{p} \in \mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$ is a $\mathbb{K}[X]$-linear combination of the rows of $\mathbf{P}$.

Note that in most cases of application, the input satisfies $\deg(\mathbf{F}) < d$; if needed, this can be ensured via fast modular reduction.

Exercise 1. Shifted reduced forms.

For each of the matrices below, and for each of the three shifts $\boldsymbol{s} = (0, 0, 0)$, $\boldsymbol{s} = (0, 5, 6)$, and $\boldsymbol{s} = (-3, -2, -2)$,

1. give the $\boldsymbol{s}$-leading matrix,
2. deduce whether the matrix is $\boldsymbol{s}$-reduced.

$$\mathbf{A} = \begin{bmatrix} 3X+4 & X^3+4X+1 & 4X^2+3 \\ 5 & 5X^2+3X+1 & 5X+3 \\ 3X^3+X^2+5X+3 & 6X+5 & 2X+1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} X^6+6X^4+X^3+X+4 & 0 & 0 \\ 5X^5+5X^4+6X^3+2X^2+6X+3 & X & 0 \\ 3X^4+5X^3+4X^2+6X+1 & 5 & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} X^3+5X^2+4X+1 & 2X+4 & 3X+5 \\ 4X^2+1 & X^2+2X+3 & X+2 \\ 2X^2+3X+2 & 4X & X^2 \end{bmatrix}$$

Exercise 2. Vector rational interpolation — some specific cases.

1. *Zero input matrix.* Assuming $\mathbf{F} = \mathbf{0}$, give a basis $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ of $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$. Verify that this basis is $\boldsymbol{s}$-reduced for any $\boldsymbol{s}$.
2. *Hermite-Padé approximation.* Assuming $\boldsymbol{\alpha} = (0, \ldots, 0) \in \mathbb{K}^d$ as well as $f_1(0) \neq 0$, prove that the following matrix:

$$\mathbf{P} = \begin{bmatrix} X^d & & & \\ h_2 & 1 & & \\ \vdots & & \ddots & \\ h_{m-1} & & & 1 \end{bmatrix} \in \mathbb{K}[X]^{m \times m},$$

   where $h_i = -f_i/f_1 \bmod X^d$, is the basis of $\mathcal{I}(\mathbf{0}, \mathbf{F})$ in Hermite form.

   Note: a square, nonsingular matrix is in Hermite form if it is lower triangular, it has monic diagonal entries, and its entries below the diagonal have degree strictly less than the diagonal entry in the same column.
3. *Case $d = 1$.* For $\boldsymbol{\alpha} = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $\mathbf{F}(\alpha)$ are nonzero, give an $\boldsymbol{s}$-reduced basis of $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$ for each of the shifts $\boldsymbol{s} = \mathbf{0}$, $\boldsymbol{s} = (2, \ldots, 2, 0)$, and $\boldsymbol{s} = (3, 0, 2, \ldots, 2)$. (We assume m sufficiently large for these shifts to make sense.)

Exercise 3. Using products of bases.

1. For $\boldsymbol{\alpha} = \mathbf{0} \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\boldsymbol{\beta} = \mathbf{0} \in \mathbb{K}^d$ (note the d):
   (a) $\mathbf{P}_1$ is a basis of $\mathcal{I}(\boldsymbol{\beta}, \mathbf{F})$;
   (b) $\mathbf{G} = (X^{-d}\mathbf{P}_1\mathbf{F}) \bmod X^d$;
   (c) $\mathbf{P}_2$ is a basis of $\mathcal{I}(\boldsymbol{\beta}, \mathbf{G})$.

   Prove that $\mathbf{P}_2\mathbf{P}_1$ is a basis of $\mathcal{I}(\boldsymbol{\alpha}, \mathbf{F})$.
2. Give an example of matrices $\mathbf{P}_1 \in \mathbb{K}[X]^{m \times m}$ and $\mathbf{P}_2 \in \mathbb{K}[X]^{m \times m}$ which are both reduced (for the shift $\mathbf{0}$) but such that $\mathbf{P}_2\mathbf{P}_1$ is not reduced.
3. ☕ Prove that if two nonsingular matrices $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{K}[X]^{m \times m}$ are such that $\mathbf{P}_1$ is $\boldsymbol{s}$-reduced and $\mathbf{P}_2$ is $\boldsymbol{t}$-reduced, for $\boldsymbol{t} = \mathrm{rdeg}_{\boldsymbol{s}}(\mathbf{P}_1)$, then the product $\mathbf{P}_2\mathbf{P}_1$ is $\boldsymbol{s}$-reduced.