Polynomial matrices
approximation and interpolation

Exercises for 2022-01-06

Recall the problem of Vector rational interpolation studied in last lecture:

**Input:**
- vector of polynomials $\mathbf{F} = [f_1 \cdots f_m]^T \in \mathbb{K}[X]^{m \times 1}$;
- points $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{K}^d$;
- shift $s = (s_1, \ldots, s_m) \in \mathbb{Z}^m$.

**Output:** matrix $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ such that $\mathbf{P}$ is $s$-reduced and the rows of $\mathbf{P}$ form a basis of the $\mathbb{K}[X]$-module $\mathcal{I}(\alpha, \mathbf{F})$.

Here, the $\mathbb{K}[X]$-submodule of $\mathbb{K}[X]^{1 \times m}$ is defined as:

$$\mathcal{I}(\alpha, \mathbf{F}) = \left\{ \mathbf{p} \in \mathbb{K}[X]^{1 \times m} \mid \mathbf{pF} = 0 \mod \prod_{1 \leq i \leq d} (X - \alpha_i) \right\}.$$ 

Recall that “$\mathbf{P}$ is a basis of $\mathcal{I}(\alpha, \mathbf{F})$” means that each row of $\mathbf{P}$ is in $\mathcal{I}(\alpha, \mathbf{F})$, and that any $\mathbf{p} \in \mathcal{I}(\alpha, \mathbf{F})$ is a $\mathbb{K}[X]$-linear combination of the rows of $\mathbf{P}$.

Note that in most cases of application, the input satisfies $\deg(\mathbf{F}) < d$; if needed, this can be ensured via fast modular reduction.

**Exercise 1. Shifted reduced forms.**

For each of the matrices below, and for each of the three shifts $s = (0, 0, 0)$, $s = (0, 5, 6)$, and $s = (-3, -2, -2)$,

1. give the $s$-leading matrix,
2. deduce whether the matrix is $s$-reduced.

$$\mathbf{A} = \begin{bmatrix} 3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\ 5 & 5X^2 + 3X + 1 & 5X + 3 \\ 3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\ 5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\ 3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\ 4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\ 2X^2 + 3X + 2 & 4X & X^2 \end{bmatrix}$$
Exercise 2. Vector rational interpolation — some specific cases.

1. Zero input matrix. Assuming $\mathbf{F} = 0$, give a basis $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ of $\mathcal{I}(\alpha, \mathbf{F})$. Verify that this basis is $s$-reduced for any $s$.

2. Hermite-Padé approximation. Assuming $\alpha = (0, \ldots, 0) \in \mathbb{K}^d$ as well as $f_1(0) \neq 0$, prove that the following matrix:

$$
\mathbf{P} = \begin{bmatrix}
X^d \\
h_2 & 1 \\
\vdots & \ddots \\
h_{m-1} & 1
\end{bmatrix} \in \mathbb{K}[X]^{m \times m},
$$

where $h_i = -f_i/f_1 \mod X^d$, is a basis of $\mathcal{I}(0, \mathbf{F})$ in Hermite form.

3. Case $d = 1$. For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $\mathbf{F}(\alpha)$ are nonzero, give an $s$-reduced basis of $\mathcal{I}(\alpha, \mathbf{F})$ for each of the shifts $s = 0$, $s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$. (We assume $m$ sufficiently large for these shifts to make sense.)


1. For $\alpha = 0 \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\beta = 0 \in \mathbb{K}^d$ (note the $d$):

(a) $\mathbf{P}_1$ is a basis of $\mathcal{I}(\beta, \mathbf{F})$;

(b) $\mathbf{G} = (X^{-d} \mathbf{P}_1 \mathbf{F}) \mod X^d$;

(c) $\mathbf{P}_2$ is a basis of $\mathcal{I}(\beta, \mathbf{G})$.

Prove that $\mathbf{P}_2 \mathbf{P}_1$ is a basis of $\mathcal{I}(\alpha, \mathbf{F})$.

2. Give an example of matrices $\mathbf{P}_1 \in \mathbb{K}[X]^{m \times m}$ and $\mathbf{P}_2 \in \mathbb{K}[X]^{m \times m}$ which are both reduced (for the shift $0$) but such that $\mathbf{P}_2 \mathbf{P}_1$ is not reduced.

3. Prove that if two nonsingular matrices $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{K}[X]^{m \times m}$ are such that $\mathbf{P}_1$ is $s$-reduced and $\mathbf{P}_2$ is $t$-reduced, for $t = \text{rdeg}_s(\mathbf{P}_1)$, then the product $\mathbf{P}_2 \mathbf{P}_1$ is $s$-reduced.