polynomial matrices:
approximation and interpolation, quasi-linear GCD

exercises and solutions

Algorithmes Efficaces en Calcul Formel
Master Parisien de Recherche en Informatique
21 December 2023
For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

\[
A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\
5 & 5X^2 + 3X + 1 & 5X + 3 \\
3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}
\]
shifted reduced forms

For each of the matrices below, and each shift $s$, 
1. give the $s$-leading matrix
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3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}$$

$s = (0, 0, 0)$

$$\text{im}_{(0,0,0)}(A) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 5 & 0 \\
3 & 0 & 0
\end{bmatrix}$$

rank 2 $\Rightarrow$ not reduced
For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
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$\text{Im}_{(0, 0, 0)}(A) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 5 & 0 \\
3 & 0 & 0
\end{bmatrix}$

rank 2 $\Rightarrow$ not reduced

$s = (0, 5, 6)$

$\text{Im}_{(0, 5, 6)}(A) = \begin{bmatrix}
0 & 1 & 4 \\
0 & 5 & 5 \\
0 & 0 & 2
\end{bmatrix}$

rank 2 $\Rightarrow$ not $s$-reduced
For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

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A = \begin{bmatrix}
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\end{bmatrix}
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\text{Im}_{(0,0,0)}(A) = \begin{bmatrix}
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\[\text{rank 2 } \Rightarrow \text{ not reduced}\]

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s = (0, 5, 6)
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\text{Im}_{(0,5,6)}(A) = \begin{bmatrix}
0 & 1 & 4 \\
0 & 5 & 5 \\
0 & 0 & 2
\end{bmatrix}
\]

\[\text{rank 2 } \Rightarrow \text{ not } s\text{-reduced}\]

\[
s = (-3, -2, -2)
\]

\[
\text{Im}_{(-3,-2,-2)}(A) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 5 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

\[\text{rank 2 } \Rightarrow \text{ not } s\text{-reduced}\]
For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1
\end{bmatrix}
\]
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
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\end{bmatrix}
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\( s = (0, 0, 0) \)

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\text{Im}_{(0,0,0)}(H) = \begin{bmatrix}
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5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

rank 1 \( \Rightarrow \) not reduced
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1 \\
\end{bmatrix}
\]

\( s = (0, 0, 0) \)
\[
\text{lm}_{(0,0,0)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0 \\
\end{bmatrix}
\]
rank 1 ⇒ not reduced

\( s = (0, 5, 6) \)
\[
\text{lm}_{(0,5,6)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
rank 3 ⇒ \( s \)-reduced
shifted reduced forms

For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$$H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1
\end{bmatrix}$$

$s = (0,0,0)$

$$\text{Im}_{(0,0,0)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}$$

rank 1 $\Rightarrow$ not reduced

$s = (0,5,6)$

$$\text{Im}_{(0,5,6)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

rank 3 $\Rightarrow$ $s$-reduced

$s = (-3,-2,-2)$

$$\text{Im}_{(-3,-2,-2)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}$$

rank 1 $\Rightarrow$ not $s$-reduced
shifted reduced forms

For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

\[
P = \begin{bmatrix}
    X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\
    4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\
    2X^2 + 3X + 2 & 4X & X^2
\end{bmatrix}
\]
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
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2X^2 + 3X + 2 & 4X & X^2 \\
\end{bmatrix}
\]

\( s = (0, 0, 0) \)

\[
\text{Im}_{(0,0,0)}(P) = \begin{bmatrix}
1 & 0 & 0 \\
4 & 1 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}
\]

rank 3 \( \Rightarrow \) reduced
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
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P = \begin{bmatrix}
X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\
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s = (0, 0, 0)
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\text{Im}_{(0,0,0)}(P) = \\
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2 & 0 & 1
\end{bmatrix}
\]
\[
\text{rank 3} \Rightarrow \text{reduced}
\]

\[
s = (0, 5, 6)
\]
\[
\text{Im}_{(0,5,6)}(P) = \\
\begin{bmatrix}
0 & 0 & 3 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
\[
\text{rank 2} \Rightarrow \text{not } s\text{-reduced}
\]
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
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P = \begin{bmatrix}
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rank 3 \( \Rightarrow \) reduced

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0 & 0 & 3 \\
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\end{bmatrix}
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rank 2 \( \Rightarrow \) not \( s \)-reduced

\[
s = (-3, -2, -2)
\]
\[
\text{im}_{(-3,-2,-2)}(P) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
rank 3 \( \Rightarrow \) \( s \)-reduced
1. **Zero input matrix.** Assuming $\mathbf{F} = 0$, give a basis $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ of $\mathcal{I}(\alpha, \mathbf{F})$. Verify that this basis is $s$-reduced for any $s$.

2. **Hermite-Padé approximation.** Assuming $\alpha = (0, \ldots, 0) \in \mathbb{K}^d$ as well as $f_1(0) \neq 0$, prove that the following matrix:

$$
\mathbf{P} = \begin{bmatrix}
\chi^d & & & \\
0 & h_2 & 1 & \\
& \ddots & \ddots & \\
& & h_{m-1} & 1
\end{bmatrix} \in \mathbb{K}[X]^{m \times m},
$$

where $h_i = -f_i/f_1 \mod \chi^d$, is the Hermite basis of $\mathcal{I}(\alpha, \mathbf{F})$.

3. **Case $d = 1$.** For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $\mathbf{F}(\alpha)$ are nonzero, give an $s$-reduced basis of $\mathcal{I}(\alpha, \mathbf{F})$ for the shifts $s = 0$, $s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$. 

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**vector rational reconstruction**
1. Zero input matrix. Assuming \( \mathbf{F} = \mathbf{0} \), give a basis \( \mathbf{P} \in \mathbb{K}[X]^{m \times m} \) of \( I(\alpha, \mathbf{F}) \). Verify that this basis is \( s \)-reduced for any \( s \).
1. **Zero input matrix.** Assuming $F = 0$, give a basis $P \in \mathbb{K}[X]^{m \times m}$ of $I(\alpha, F)$. Verify that this basis is $s$-reduced for any $s$.

Consider the identity matrix

$$
P = \begin{bmatrix} 1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \end{bmatrix} \in \mathbb{K}[X]^{m \times m}.
$$

Then $P$ is a basis of $\mathbb{K}[X]^{1 \times m}$, which is $I(\alpha, F)$ when $F = 0$.

The $s$-leading matrix of $P$ is the identity in $\mathbb{K}^{m \times m}$ for any shift $s \in \mathbb{Z}^m$. 
2. Hermite-Padé approximation. Assuming $\alpha = (0, \ldots, 0) \in K^d$ as well as $f_1(0) \neq 0$, prove that the following matrix:

$$P = \begin{bmatrix}
X^d & 1 \\
h_2 & 1 \\
\vdots & \ddots & \ddots \\
h_{m-1} & 1
\end{bmatrix} \in K[X]^{m \times m},$$

where $h_i = -f_i/f_1 \mod X^d$, is the Hermite basis of $I(\alpha, F)$. 

vector rational reconstruction
2. **Hermite-Padé approximation.** Assuming \( \alpha = (0, \ldots, 0) \in \mathbb{K}^d \) as well as \( f_1(0) \neq 0 \), prove that the following matrix:

\[
P = \begin{bmatrix} X^d & 1 \\ h_2 & 1 \\ \vdots & \vdots \\ h_{m-1} & 1 \end{bmatrix} \in \mathbb{K}[X]^{m \times m},
\]

where \( h_i = -f_i/f_1 \mod X^d \), is the Hermite basis of \( J(\alpha, F) \).

The matrix \( P \) is well defined since \( f_1 \) is invertible modulo \( X^d \).
It is in Hermite form: lower triangular, monic diagonal entries, and entries below the diagonal have degree strictly less than the diagonal entry in the same column.
By construction, \( PF = 0 \mod X^d \).

Let \( H \) be the basis of \( J(0, F) \) in Hermite form.
Then \( P = UH \) for some nonsingular \( U \).
The first diagonal entry \( h_{11} \) satisfies \( h_{11}f_1 = 0 \mod X^d \), hence \( h_{11} \) is a multiple of \( X^d \).
Therefore \( \deg(\det(H)) \geq d = \deg(\det(P)) = \deg(\det(U)) + \deg(\det(H)) \)
Hence \( U \) is unimodular (and in fact \( H = P \) by uniqueness).
3. **Case** $d = 1$. For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $\mathbf{F}(\alpha)$ are nonzero, give an $s$-reduced basis of $\mathcal{I}(\alpha, \mathbf{F})$ for the shifts $s = \mathbf{0}, s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$. 

**vector rational reconstruction**
3. Case $d = 1$. For $\alpha = (\alpha) \in K^1$ (i.e. $d = 1$), and assuming all entries of $F(\alpha)$ are nonzero, give an $s$-reduced basis of $I(\alpha, F)$ for the shifts $s = 0, s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$.

We write $g_i = f_i(\alpha)$ for $1 \leq i \leq d$.

For $s = (0, \ldots, 0)$, take $P = \begin{bmatrix} X - \alpha & 1 \\ -g_2/g_1 & 1 \\ \vdots & \vdots \\ -g_m/g_1 & 1 \end{bmatrix}$

For $s = (2, \ldots, 2, 0)$, take $P = \begin{bmatrix} 1 & -g_1/g_m \\ \vdots & \vdots \\ 1 - g_{m-1}/g_m & 1 - g_{m-1}/g_m \\ X - \alpha & 1 \end{bmatrix}$

For $s = (3, 0, 2, \ldots, 2)$, take $P = \begin{bmatrix} 1 & -g_1/g_2 \\ X - \alpha & 1 \\ -g_3/g_2 & 1 \\ \vdots & \vdots \\ -g_m/g_2 & 1 \end{bmatrix}$
products of bases

1. For $\alpha = 0 \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\beta = 0 \in \mathbb{K}^d$ (note the $d$):
   - $P_1$ is a basis of $I(\beta, F)$;
   - $G = (X^{-d}P_1F) \mod X^d$;
   - $P_2$ is a basis of $I(\beta, G)$.
   Prove that $P_2P_1$ is a basis of $I(\alpha, F)$.

2. Give an example of matrices $P_1$ and $P_2$ in $\mathbb{K}[X]^{m \times m}$ which are both reduced but such that $P_2P_1$ is not reduced.

3. Prove that if two nonsingular matrices $P_1, P_2 \in \mathbb{K}[X]^{m \times m}$ are such that $P_1$ is $s$-reduced and $P_2$ is $t$-reduced, for $t = \text{rdeg}_s(P_1)$, then the product $P_2P_1$ is $s$-reduced.
products of bases

1. For $\alpha = 0 \in K^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\beta = 0 \in K^d$ (note the $d$):
   - $P_1$ is a basis of $J(\beta, F)$;
   - $G = (X^{-d}P_1F) \mod X^d$;
   - $P_2$ is a basis of $J(\beta, G)$.
Prove that $P_2P_1$ is a basis of $J(\alpha, F)$. 
products of bases

1. For $\alpha = 0 \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\beta = 0 \in \mathbb{K}^d$ (note the $d$):
   - $P_1$ is a basis of $\mathcal{I}(\beta, F)$;
   - $G = (X^{-d}P_1F) \mod X^d$;
   - $P_2$ is a basis of $\mathcal{I}(\beta, G)$.

Prove that $P_2P_1$ is a basis of $\mathcal{I}(\alpha, F)$.

A row $p$ of $P_2P_1$ has the form $p = qP_1$ for some row $q$ of $P_2$. Working modulo $X^{2d}$, we have $X^dG = (P_1F) \mod X^{2d}$.

Hence $pF = qP_1F = q(X^dG) = X^d(qG) \mod X^{2d}$.
Since $q$ is a row of $P_2$, $qG = 0 \mod X^d$, and we get $pF = 0 \mod X^{2d}$.
Hence $p \in \mathcal{I}(\alpha, F)$.

Now let $p \in \mathcal{I}(\alpha, F)$, i.e. $pF = 0 \mod X^{2d}$.
This implies $pF = 0 \mod X^d$ hence $p = \lambda P_1$ for some $\lambda \in \mathbb{K}[X]^{1 \times m}$.
But then $\lambda P_1F = 0 \mod X^{2d}$, hence $\lambda G = X^{-d}(\lambda P_1F) \mod X^d = 0 \mod X^d$.
Thus $\lambda = \mu P_2$ for some $\mu \in \mathbb{K}[X]^{1 \times m}$.
This yields $p = \mu P_2P_1$, i.e. $p$ is a $\mathbb{K}[X]$-linear combination of the rows of $P_2P_1$. 
2. Give an example of matrices $P_1$ and $P_2$ in $K[X]^{m \times m}$ which are both reduced but such that $P_2P_1$ is not reduced.
2. Give an example of matrices $P_1$ and $P_2$ in $\mathbb{K}[X]^{m \times m}$ which are both reduced but such that $P_2 P_1$ is not reduced.

Consider the reduced matrix seen above,

$$P_2 = \begin{bmatrix} X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\ 4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\ 2X^2 + 3X + 2 & 4X & X^2 \end{bmatrix}$$

And consider the reduced matrix

$$P_1 = \begin{bmatrix} X \\ 1 \\ 1 \end{bmatrix}$$

The following matrix is not reduced:

$$P_2 P_1 = \begin{bmatrix} X^4 + 5X^3 + 4X^2 + X & 2X + 4 & 3X + 5 \\ 4X^3 + X & X^2 + 2X + 3 & X + 2 \\ 2X^3 + 3X^2 + 2X & 4X & X^2 \end{bmatrix}$$
3. Prove that if two nonsingular matrices $P_1, P_2 \in \mathbb{K}[X]^{m \times m}$ are such that $P_1$ is $s$-reduced and $P_2$ is $t$-reduced, for $t = \text{rdeg}_s(P_1)$, then the product $P_2 P_1$ is $s$-reduced.
3. Prove that if two nonsingular matrices $P_1, P_2 \in \mathbb{K}[X]^{m \times m}$ are such that $P_1$ is $s$-reduced and $P_2$ is $t$-reduced, for $t = \text{rdeg}_s(P_1)$, then the product $P_2P_1$ is $s$-reduced.

Let $d = \text{rdeg}_t(P_2)$.

Then $d = \text{rdeg}_s(P_2P_1)$ by the predictable degree property.

Using $X^{-d}P_2P_1X^s = X^{-d}P_2X^tX^{-t}P_1X^s$, we obtain that $\text{lm}_s(P_2P_1) = \text{lm}_t(P_2)\text{lm}_s(P_1)$.

By assumption, $\text{lm}_t(P_2)$ and $\text{lm}_s(P_1)$ are invertible.

Therefore $\text{lm}_s(P_2P_1)$ is invertible as well, i.e. $P_2P_1$ is $s$-reduced.