polynomial matrices:
approximation and interpolation, quasi-linear GCD

exercises and solutions

Algorithmes Efficaces en Calcul Formel
Master Parisien de Recherche en Informatique
13 December 2022
shifted reduced forms

For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$$A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\
5 & 5X^2 + 3X + 1 & 5X + 3 \\
3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}$$
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\
5 & 5X^2 + 3X + 1 & 5X + 3 \\
3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}
\]

\( s = (0, 0, 0) \)

\[
\text{Im}_{(0,0,0)}(A) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 5 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

rank 2 \( \Rightarrow \) not reduced
For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\
5 & 5X^2 + 3X + 1 & 5X + 3 \\
3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}
\]

\( s = (0, 0, 0) \)

\[
\text{Im}_{(0,0,0)}(A) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 5 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

\( \text{rank } 2 \Rightarrow \text{not reduced} \)

\( s = (0, 5, 6) \)

\[
\text{Im}_{(0,5,6)}(A) = \begin{bmatrix}
0 & 1 & 4 \\
0 & 5 & 5 \\
0 & 0 & 2
\end{bmatrix}
\]

\( \text{rank } 2 \Rightarrow \text{not } s\text{-reduced} \)
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X^2 + 3 \\
5 & 5X^2 + 3X + 1 & 5X + 3 \\
3X^3 + X^2 + 5X + 3 & 6X + 5 & 2X + 1
\end{bmatrix}
\]

\[s = (0, 0, 0)\]
\[\text{Im}_{(0,0,0)}(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 5 & 0 \\ 3 & 0 & 0 \end{bmatrix}\]
rank 2 ⇒ not reduced

\[s = (0, 5, 6)\]
\[\text{Im}_{(0,5,6)}(A) = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}\]
rank 2 ⇒ not \( s \)-reduced

\[s = (−3, −2, −2)\]
\[\text{Im}_{(−3,−2,−2)}(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 5 & 0 \\ 3 & 0 & 0 \end{bmatrix}\]
rank 2 ⇒ not \( s \)-reduced
For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$$H = \begin{bmatrix} X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\ 5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\ 3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1 \end{bmatrix}$$
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1
\end{bmatrix}
\]

\( s = (0, 0, 0) \)

\[
\text{Im}_{(0,0,0)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

rank 1 \( \Rightarrow \) not reduced
shifted reduced forms

For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$$H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1
\end{bmatrix}$$

$s = (0, 0, 0)$

$$\text{Im}_{(0,0,0)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}$$

rank 1 $\Rightarrow$ not reduced

$s = (0, 5, 6)$

$$\text{Im}_{(0,5,6)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

rank 3 $\Rightarrow$ $s$-reduced
shifted reduced forms

For each of the matrices below, and each shift \( s \),
1. give the \( s \)-leading matrix
2. deduce whether the matrix is \( s \)-reduced

\[
H = \begin{bmatrix}
X^6 + 6X^4 + X^3 + X + 4 & 0 & 0 \\
5X^5 + 5X^4 + 6X^3 + 2X^2 + 6X + 3 & X & 0 \\
3X^4 + 5X^3 + 4X^2 + 6X + 1 & 5 & 1
\end{bmatrix}
\]

\( s = (0, 0, 0) \)
\[
\text{Im}_{(0,0,0)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]
rank 1 \( \Rightarrow \) not reduced

\( s = (0, 5, 6) \)
\[
\text{Im}_{(0,5,6)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
rank 3 \( \Rightarrow \) \( s \)-reduced

\( s = (-3, -2, -2) \)
\[
\text{Im}_{(-3,-2,-2)}(H) = \begin{bmatrix}
1 & 0 & 0 \\
5 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]
rank 1 \( \Rightarrow \) not \( s \)-reduced
For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$A = \begin{bmatrix}
3X + 4 & X^3 + 4X + 1 & 4X + 3 \\
X^3 + 4X + 1 & 2X + 4 & 3X + 5 \\
2X^2 + 3X + 2 & X^2 + 2X + 3 & X + 2
\end{bmatrix}$

$s = (0, 0, 0)$

$lm(0,0,0) \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \ 0 & 5 & 0 \ 3 & 0 & 0 \end{bmatrix}$

$\text{rank } 2 \Rightarrow \text{not reduced}$

$s = (0, 5, 6)$

$lm(0,5,6) \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \ 0 & 5 & 5 \ 0 & 0 & 2 \end{bmatrix}$

$\text{rank } 2 \Rightarrow \text{not } s\text{-reduced}$

$s = (-3, -2, -2)$

$lm(-3,-2,-2) \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \ 0 & 5 & 0 \ 3 & 0 & 0 \end{bmatrix}$

$\text{rank } 2 \Rightarrow \text{not } s\text{-reduced}$

$H = \begin{bmatrix}
X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\
4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\
2X^2 + 3X + 2 & 4X & X^2
\end{bmatrix}$

$s = (0, 0, 0)$

$lm(0,0,0) \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ 5 & 0 & 0 \ 3 & 0 & 0 \end{bmatrix}$

$\text{rank } 1 \Rightarrow \text{not reduced}$

$s = (0, 5, 6)$

$lm(0,5,6) \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{bmatrix}$

$\text{rank } 3 \Rightarrow s\text{-reduced}$

$s = (-3, -2, -2)$

$lm(-3,-2,-2) \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ 5 & 0 & 0 \ 3 & 0 & 0 \end{bmatrix}$

$\text{rank } 1 \Rightarrow \text{not } s\text{-reduced}$

$P = \begin{bmatrix}
X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\
4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\
2X^2 + 3X + 2 & 4X & X^2
\end{bmatrix}$

$s = (0, 0, 0)$

$lm(0,0,0) \begin{bmatrix} P \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ 4 & 1 & 0 \ 2 & 0 & 1 \end{bmatrix}$

$\text{rank } 3 \Rightarrow \text{reduced}$

$s = (0, 5, 6)$

$lm(0,5,6) \begin{bmatrix} P \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \ 0 & 1 & 1 \ 0 & 0 & 1 \end{bmatrix}$

$\text{rank } 2 \Rightarrow \text{not } s\text{-reduced}$

$s = (-3, -2, -2)$

$lm(-3,-2,-2) \begin{bmatrix} P \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{bmatrix}$

$\text{rank } 3 \Rightarrow s\text{-reduced}$
For each of the matrices below, and each shift $s$, 
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

$A = \begin{bmatrix}
3X^2 + 4 & X^2 + 1 & 2X + 4 \\
2X^2 + 3X + 2 & X^2 + 2X + 3 & X + 2 \\
\end{bmatrix}$

$s = (0, 0, 0)$

$\text{Im}_{(0,0,0)}(P) = \begin{bmatrix}
1 & 0 & 0 \\
4 & 1 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}$

rank 3 $\Rightarrow$ reduced
shifted reduced forms

For each of the matrices below, and each shift $s$,
1. give the $s$-leading matrix
2. deduce whether the matrix is $s$-reduced

\[ A = \begin{bmatrix} 3X + 4 & 4X + 1 & 1 \\ 4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\ 2X^2 + 3X + 2 & 4X & X^2 \end{bmatrix} \]

$s = (0, 0, 0) \quad \text{Im}_{(0,0,0)}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

rank 3 $\Rightarrow$ reduced

$s = (0, 5, 6) \quad \text{Im}_{(0,5,6)}(A) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

rank 2 $\Rightarrow$ not $s$-reduced
For each of the matrices below, and each shift $s$, 
1. give the $s$-leading matrix 
2. deduce whether the matrix is $s$-reduced

\[ P = \begin{bmatrix} 
   X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\
   4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\
   2X^2 + 3X + 2 & 4X & X^2
\end{bmatrix} \]

$s = (0, 0, 0)$ 
\[ \text{Im}_{(0,0,0)}(P) = \begin{bmatrix} 
   1 & 0 & 0 \\
   4 & 1 & 0 \\
   2 & 0 & 1
\end{bmatrix} \]
rank 3 $\Rightarrow$ reduced

$s = (0, 5, 6)$ 
\[ \text{Im}_{(0,5,6)}(P) = \begin{bmatrix} 
   0 & 0 & 3 \\
   0 & 1 & 1 \\
   0 & 0 & 1
\end{bmatrix} \]
rank 2 $\Rightarrow$ not $s$-reduced

$s = (-3, -2, -2)$ 
\[ \text{Im}_{(-3,-2,-2)}(P) = \begin{bmatrix} 
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
\end{bmatrix} \]
rank 3 $\Rightarrow$ $s$-reduced
1. **Zero input matrix.** Assuming $\mathbf{F} = 0$, give a basis $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ of $\mathcal{I}(\alpha, \mathbf{F})$. Verify that this basis is $s$-reduced for any $s$.

2. **Hermite-Padé approximation.** Assuming $\alpha = (0, \ldots, 0) \in \mathbb{K}^d$ as well as $f_1(0) \neq 0$, prove that the following matrix:

   \[
   \mathbf{P} = \begin{bmatrix}
   X^d \\
   h_2 & 1 \\
   \vdots & \ddots \\
   h_{m-1} & 1
   \end{bmatrix} \in \mathbb{K}[X]^{m \times m},
   \]

   where $h_i = -f_i/f_1 \mod X^d$, is the Hermite basis of $\mathcal{I}(\alpha, \mathbf{F})$.

3. **Case $d = 1$.** For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $\mathbf{F}(\alpha)$ are nonzero, give an $s$-reduced basis of $\mathcal{I}(\alpha, \mathbf{F})$ for the shifts $s = 0$, $s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$. 

---

**vector rational reconstruction**
1. Zero input matrix. Assuming $F = 0$, give a basis $P \in \mathbb{K}[X]^{m \times m}$ of $J(\alpha, F)$. Verify that this basis is $s$-reduced for any $s$. 

vector rational reconstruction
1. Zero input matrix. Assuming \( \mathbf{F} = 0 \), give a basis \( \mathbf{P} \in \mathbb{K}[X]^{m \times m} \) of \( J(\alpha, \mathbf{F}) \). Verify that this basis is \( s \)-reduced for any \( s \).

Consider the identity matrix

\[
\mathbf{P} = \begin{bmatrix}
1 \\
. \\
. \\
. \\
1
\end{bmatrix} \in \mathbb{K}[X]^{m \times m}.
\]

Then \( \mathbf{P} \) is a basis of \( \mathbb{K}[X]^{1 \times m} \), which is \( J(\alpha, \mathbf{F}) \) when \( \mathbf{F} = 0 \).

The \( s \)-leading matrix of \( \mathbf{P} \) is the identity in \( \mathbb{K}^{m \times m} \) for any shift \( s \in \mathbb{Z}^m \).
2. Hermite-Padé approximation. Assuming \( \alpha = (0, \ldots, 0) \in \mathbb{K}^d \) as well as \( f_1(0) \neq 0 \), prove that the following matrix:

\[
P = \begin{bmatrix}
X^d & 1 \\
h_2 & 1 \\
\vdots & \ddots \\
h_{m-1} & 1
\end{bmatrix} \in \mathbb{K}[X]^{m \times m},
\]

where \( h_i = -f_i/f_1 \mod X^d \), is the Hermite basis of \( J(\alpha, F) \).
2. *Hermite-Padé approximation*. Assuming \( \alpha = (0, \ldots, 0) \in \mathbb{K}^d \) as well as \( f_1(0) \neq 0 \), prove that the following matrix:

\[
P = \begin{bmatrix}
X^d & 1 \\
h_2 & 1 \\
\vdots & \ddots \\
h_{m-1} & 1
\end{bmatrix} \in \mathbb{K}[X]^{m \times m},
\]

where \( h_i = -f_i/f_1 \mod X^d \), is the Hermite basis of \( J(\alpha, F) \).

The matrix \( P \) is well defined since \( f_1 \) is invertible modulo \( X^d \).
It is in Hermite form: lower triangular, monic diagonal entries, and entries below the diagonal have degree strictly less than the diagonal entry in the same column.
By construction, \( PF = 0 \mod X^d \).

Let \( H \) be the basis of \( J(0, F) \) in Hermite form.
Then \( P = UH \) for some nonsingular \( U \).
The first diagonal entry \( h_{11} \) satisfies \( h_{11} f_1 = 0 \mod X^d \),
hence \( h_{11} \) is a multiple of \( X^d \).
Therefore \( \deg(\det(H)) \geq d = \deg(\det(P)) = \deg(\det(U)) + \deg(\det(H)) \)
Hence \( U \) is unimodular (and in fact \( H = P \) by uniqueness).
3. Case $d = 1$. For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $F(\alpha)$ are nonzero, give an $s$-reduced basis of $J(\alpha, F)$ for the shifts $s = 0$, $s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$. 
3. Case $d = 1$. For $\alpha = (\alpha) \in \mathbb{K}^1$ (i.e. $d = 1$), and assuming all entries of $F(\alpha)$ are nonzero, give an $s$-reduced basis of $I(\alpha, F)$ for the shifts $s = 0$, $s = (2, \ldots, 2, 0)$, and $s = (3, 0, 2, \ldots, 2)$.

We write $g_i = f_i(\alpha)$ for $1 \leq i \leq d$.

For $s = (0, \ldots, 0)$, take $P = \begin{bmatrix} X-\alpha & 1 \\ -g_2/g_1 & 1 \\ \vdots & \vdots \\ -g_m/g_1 & 1 \end{bmatrix}$.

For $s = (2, \ldots, 2, 0)$, take $P = \begin{bmatrix} 1 & -g_1/g_m \\ \vdots & \vdots \\ 1 - g_{m-1}/g_m & X-\alpha \end{bmatrix}$.

For $s = (3, 0, 2, \ldots, 2)$, take $P = \begin{bmatrix} 1 & -g_1/g_2 & 1 \\ X-\alpha & -g_3/g_2 & 1 \\ \vdots & \vdots & \vdots \\ -g_m/g_2 & 1 \end{bmatrix}$. 
products of bases

1. For \( \alpha = 0 \in K^{2d} \) (Hermite-Padé approximation at order 2d), assume the following, where \( \beta = 0 \in K^d \) (note the d):
   - \( P_1 \) is a basis of \( J(\beta, F) \);
   - \( G = (X^{-d}P_1F) \mod X^d \);
   - \( P_2 \) is a basis of \( J(\beta, G) \).

   Prove that \( P_2P_1 \) is a basis of \( J(\alpha, F) \).

2. Give an example of matrices \( P_1 \) and \( P_2 \) in \( K[X]^{m \times m} \) which are both reduced but such that \( P_2P_1 \) is not reduced.

3. Prove that if two nonsingular matrices \( P_1, P_2 \in K[X]^{m \times m} \) are such that \( P_1 \) is \( s \)-reduced and \( P_2 \) is \( t \)-reduced, for \( t = \text{rdeg}_s(P_1) \), then the product \( P_2P_1 \) is \( s \)-reduced.
1. For $\alpha = 0 \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order $2d$), assume the following, where $\beta = 0 \in \mathbb{K}^d$ (note the $d$):

- $P_1$ is a basis of $J(\beta, F)$;
- $G = (X^{-d}P_1F) \mod X^d$;
- $P_2$ is a basis of $J(\beta, G)$.

Prove that $P_2P_1$ is a basis of $J(\alpha, F)$.
products of bases

1. For $\alpha = 0 \in \mathbb{K}^{2d}$ (Hermite-Padé approximation at order 2d), assume the following, where $\beta = 0 \in \mathbb{K}^d$ (note the d):
   - $P_1$ is a basis of $I(\beta, F)$;
   - $G = (X^{-d}P_1F) \mod X^d$;
   - $P_2$ is a basis of $I(\beta, G)$.

Prove that $P_2P_1$ is a basis of $I(\alpha, F)$.

A row $p$ of $P_2P_1$ has the form $p = qP_1$ for some row $q$ of $P_2$.
Working modulo $X^{2d}$, we have $X^dG = (P_1F) \mod X^{2d}$.
Hence $pF = qP_1F = q(X^dG) = X^d(qG) \mod X^{2d}$.
Since $q$ is a row of $P_2$, $qG = 0 \mod X^d$, and we get $pF = 0 \mod X^{2d}$.
Hence $p \in I(\alpha, F)$.

Now let $p \in I(\alpha, F)$, i.e. $pF = 0 \mod X^{2d}$.
This implies $pF = 0 \mod X^d$ hence $p = \lambda P_1$ for some $\lambda \in \mathbb{K}[X]^{1 \times m}$.
But then $\lambda P_1F = 0 \mod X^{2d}$, hence $\lambda G = X^{-d}(\lambda P_1F) \mod X^d = 0 \mod X^d$.
Thus $\lambda = \mu P_2$ for some $\mu \in \mathbb{K}[X]^{1 \times m}$.
This yields $p = \mu P_2P_1$, i.e. $p$ is a $\mathbb{K}[X]$-linear combination of the rows of $P_2P_1$. 
2. Give an example of matrices $P_1$ and $P_2$ in $K[X]^{m \times m}$ which are both reduced but such that $P_2 P_1$ is not reduced.
products of bases

2. Give an example of matrices $P_1$ and $P_2$ in $\mathbb{K}[X]^{m \times m}$ which are both reduced but such that $P_2P_1$ is not reduced.

Consider the reduced matrix seen above,

$$P_2 = \begin{bmatrix} X^3 + 5X^2 + 4X + 1 & 2X + 4 & 3X + 5 \\ 4X^2 + 1 & X^2 + 2X + 3 & X + 2 \\ 2X^2 + 3X + 2 & 4X & X^2 \end{bmatrix}$$

And consider the reduced matrix

$$P_1 = \begin{bmatrix} X \\ 1 \\ 1 \end{bmatrix}$$

The the following matrix is not reduced:

$$P_2P_1 = \begin{bmatrix} X^4 + 5X^3 + 4X^2 + X & 2X + 4 & 3X + 5 \\ 4X^3 + X & X^2 + 2X + 3 & X + 2 \\ 2X^3 + 3X^2 + 2X & 4X & X^2 \end{bmatrix}$$
3. Prove that if two nonsingular matrices $P_1, P_2 \in \mathbb{K}[X]^{m \times m}$ are such that $P_1$ is $s$-reduced and $P_2$ is $t$-reduced, for $t = rdeg_s(P_1)$, then the product $P_2 P_1$ is $s$-reduced.
products of bases

3. ⚫ Prove that if two nonsingular matrices $P_1, P_2 \in \mathbb{K}[X]^{m \times m}$ are such that $P_1$ is $s$-reduced and $P_2$ is $t$-reduced, for $t = \text{rdeg}_s(P_1)$, then the product $P_2P_1$ is $s$-reduced.

Let $d = \text{rdeg}_t(P_2)$.
Then $d = \text{rdeg}_s(P_2P_1)$ by the predictable degree property.
Using $X^{-d}P_2P_1X^s = X^{-d}P_2X^tX^{-t}P_1X^s$,
we obtain that $\text{Im}_s(P_2P_1) = \text{Im}_t(P_2)\text{Im}_s(P_1)$.
By assumption, $\text{Im}_t(P_2)$ and $\text{Im}_s(P_1)$ are invertible.
Therefore $\text{Im}_s(P_2P_1)$ is invertible as well, i.e. $P_2P_1$ is $s$-reduced.