Generic bivariate multi-point evaluation, interpolation and modular composition with precomputation

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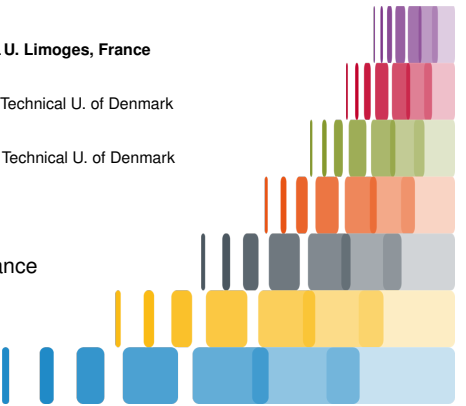

Outline

- Problem and result
- Approach: degree reshaping
- Application to composition, evaluation, interpolation

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Computing with univariate polynomials

$p, f, a, b, \ldots \in \mathbb{K}[x]$, degree n, for some base field $\mathbb{K}$

quasi-linear algorithms are known for most basic operations

[v.z.Gathen - Gerhard, Modern Computer Algebra, 3rd ed. 2013]

- addition $f + g$, multiplication $f * g$
- division with remainder $f = ag + b$
- truncated inverse $f^{-1} \bmod x^n$
- extended GCD $af + bg = \gcd(f, g)$
- multipoint evaluation $f \mapsto f(x_1), \ldots, f(x_n)$
- interpolation $f(x_1), \ldots, f(x_n) \mapsto f$

Modular composition $b = p(a) \bmod f$

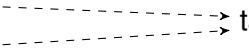
Notable exception: best known exponent $\frac{\omega+1}{2}$ [Brent - Kung 1978]

Work in progress: exponent $\approx \frac{\omega+2}{3}$, randomized [Neiger - Salvy - Schost - Villard]

Efficient? Fast?

Measuring efficiency/speed:

- low complexity bound
- low execution time
- low memory usage, low power consumption, ...



Complexity = algebraic cost (count number of operations in $\mathbb{K}$)

- standard model for algebraic computations
- accurate for finite fields $\mathbb{K} = \mathbb{F}_p$ (e.g. coding theory, crypto, ...)
- ignores coefficient growth (e.g. when working over $\mathbb{K} = \mathbb{Q}$)

Quasi-linear: $\tilde{O}(n)$ operations in $\mathbb{K}$

the holy grail of computer algebraists... or the start of the fight with logarithmic factors

Computing with bivariate polynomials

$p, q, r, \ldots \in \mathbb{K}[x, y]$, bi-degree (d_x, d_y) with $d_x d_y = n$.

quasi-linear algorithms are known for some basic operations,

generally with specific input/output requirements (sparsity, genericity, ...)

- addition $f + g$, multiplication $f * g$ (Kronecker substitution / evaluation-interpolation)
- division with remainder / reduction modulo an ideal
  - w.r.t. autoreduced family [v.d.Hoeven 2015] — sparse
  - modulo two polynomials [v.d.Hoeven - Larrieu 2019] — generic
- multiplication modulo a triangular set
  - using homotopy [Bostan - Chowdhury - v.d.Hoeven - Schost 2011] — sparse
  - reduction to modular composition [Poteaux - Schost 2013]
- multipoint evaluation & interpolation $p \longleftrightarrow p(x_1, y_1), \ldots, p(x_n, y_n)$
  - interpolation [Prony 1795] [Ben-or - Tiwari 1988] — sparse
  - eval./interp.: points&monomials on a grid [Pan 1994] [v.d.Hoeven - Schost 2013] — grid
  - evaluation: reduction to modular composition [Nüsken - Ziegler 2004]

Evaluation and composition

Bivariate multipoint evaluation

Input: $\{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{K}^2$
$p(x, y)$ of degree $< (d_x, d_y)$ with $d_x d_y \in O(n)$
Output: $z_1 = p(x_1, y_1), \ldots, z_n = p(x_n, y_n)$

Bivariate modular composition

Input: $f(x), a(x)$ with $\deg(a) < \deg(f) = n$
$p(x, y)$ of degree $< (d_x, d_y)$ with $d_x d_y \in O(n)$
Output: $b(x) = p(x, a(x)) \bmod f(x)$

[Brent - Kung 1978] $\tilde{O}(n^{\frac{\omega+1}{2}})$ for $d_x = 1$ [Nüsken - Ziegler 2004] $\tilde{O}(d_x d_y^{\frac{\omega+1}{2}})$

Evaluation and composition

Bivariate multipoint evaluation

Input: $\{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{K}^2$ with $x_1, \ldots, x_n$ distinct
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Ideal $\mathcal{I} = \langle f(x), y - a(x) \rangle$

Reduction: [Nüsken - Ziegler 2004]
· $f(x) = (x - x_1) \cdots (x - x_n)$
· $a(x)$ the interpolant $a(x_i) = y_i$

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Interpolation and inverse composition

Bivariate interpolation

Input: $\{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{K}^2$
$z_1, \ldots, z_n \in \mathbb{K}$ and degree bound d_y
Output: $p(x, y)$ with $z_i = p(x_i, y_i)$ and degree $< (n/d_y, d_y)$

Bivariate inverse modular composition

Input: $f(x), a(x)$ with $\deg(a) < \deg(f) = n$
$b(x)$ of degree $< n$ and degree bound d_y
Output: $p(x, y)$ with $b(x) = p(x, a(x)) \bmod f(x)$ and deg $< (n/d_y, d_y)$

$\tilde{O}(d_y^{\omega-1} n)$ using [Storjohann 2003] + [Neiger 2016] $\rightsquigarrow \tilde{O}(n^{\frac{\omega+1}{2}})$ if $d_y \approx \sqrt{n}$

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Contribution

Precomputation from f, a or from $(x_i, y_i)_i$

Reshaper polynomial $r_1(x, y), \ldots, r_k(x, y)$ with $k \approx \log_{3/2}(n)$
$\rightsquigarrow$ fast remainders modulo $\mathcal{I}$ with specific degree shapes

Cost of precomputing reshapers: naive exponent ω, prospective exponent $\frac{\omega+1}{2}$

Online stage (evaluation/composition/interpolation)

cost $\tilde{O}(n + D)$, quasi-linear $\tilde{O}(n)$ for generic preinput

D = total size of reshapers $= \sum_i \deg_x(r_i) \deg_y(r_i)$
Generically each reshaper has size $\approx n$, and $D \in O(n \log(n))$

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Van der Hoeven and Lecerf obtained a similar result [Technical Report, 2020]
· quasi-linear multivariate evaluation and interpolation
· with precomputation and genericity assumption
· different approach based on generalizing the subproduct trees
· does not seem easily amenable to performing modular composition

Genericity and precomputations

Genericity assumptions often used recently for bivariate polynomials

· resultant of bivariate polynomials [Villard 2018]
· Gröbner basis and reduction for bivariate ideals [v.d.Hoeven - Larrieu 2019]
· modular composition and related problems [Neiger - Salvy - Schost - Villard, in progress]

- used to predict and leverage degree shapes of bases of ideals
- hoped next step: genericity removed via randomization

Situations allowing precomputations arise naturally

- encoding of some algebraic geometry error correcting codes [Miura 1993]
  (similar to Reed-Solomon encoding, but with bivariate polynomials)
- subroutine in distinct-degree factorization
- endless source of speed-ups in implementations

Precomputations in implementations

Shoup's NTL C++ library:

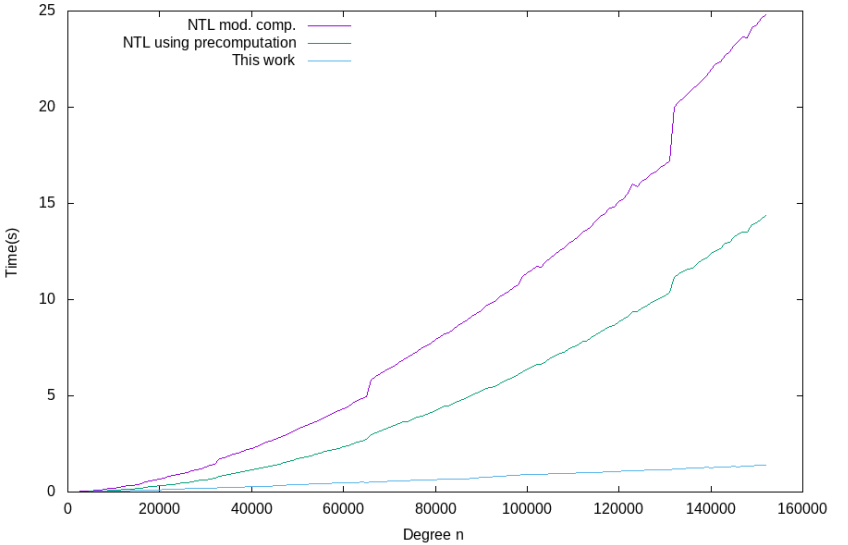
Excerpts from the documentation on univariate polynomials

```
/*************************************************************\
            Modular Arithmetic with Pre-Conditioning

If you need to do a lot of arithmetic modulo a fixed f, build
zz_pXModulus F for f.  This pre-computes information about f
that speeds up subsequent computations.

/*************************************************************\
                    More Pre-Conditioning

If you need to compute a * b % f for a fixed b, but for many
a's, it is much more efficient to first build a zz_pXMultiplier
B for b, and then use the MulMod routine below.

/*************************************************************\
    Faster Composition and Projection with Pre-Conditioning

If a single h is going to be used with many g's then you should
build a zz_pXArgument for h, and use the compose routine below.
```

Practicality of the new technique

Comparison of timings: modular composition, excluding precomputation

⇝ **significant speed-up**, including for small degrees



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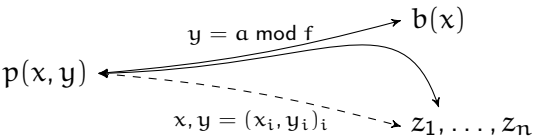
Reshaping degrees modulo the ideal

Our problems ask to compute remainders modulo $\mathcal{I} = \langle f(x), y - a(x) \rangle$



Example: $p(x, y)$ of degree $(0, n) \xleftrightarrow{\bmod \mathcal{I}} b(x)$ of degree $(n, 0)$
$\rightsquigarrow$ opposite degree shapes, difficult one-step transformation

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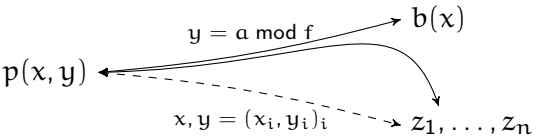

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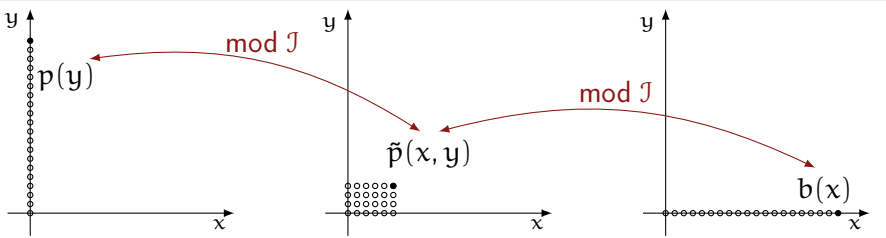
we can use k intermediate transformations, for small k



Intermediate reshaping transformation

For input degree bound d_y, define reshaper $r(x, y)$:

- $y^{\frac{2}{3}d_y} - r(x, y)$ belongs to $\mathcal{I}$
- $r(x, y)$ has y-degree $< \frac{1}{3}d_y$, and minimal x-degree

$\rightsquigarrow$ $r(x, y)$ exists since $y - a(x) \in \mathcal{I}$
$\rightsquigarrow$ it has x-degree $< 3n/d_y$ generically

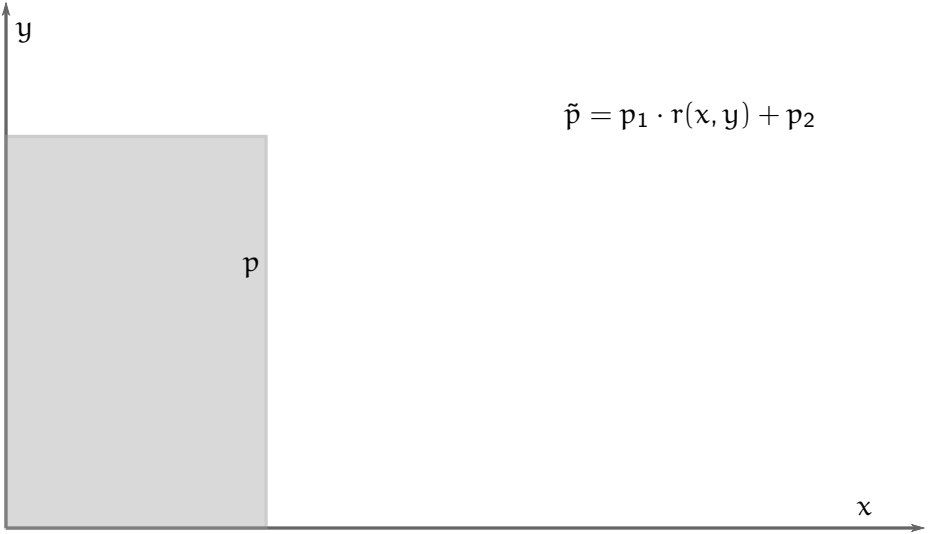
Remainder = bivariate multiplication, and degrees are controlled

For input polynomial $p(x, y)$ of degree $< (d_x, d_y)$,

- write $p = p_1 y^{\frac{2}{3}d_y} + p_2$
- then $p \xrightarrow{\bmod \mathcal{I}} \tilde{p} = p_1 \cdot r(x, y) + p_2$ (multiplication, quasi-linear)
- $\tilde{p}$ has y-degree $< \frac{2}{3}d_y$, and x-degree $\leqslant d_x + \deg_x(r)$

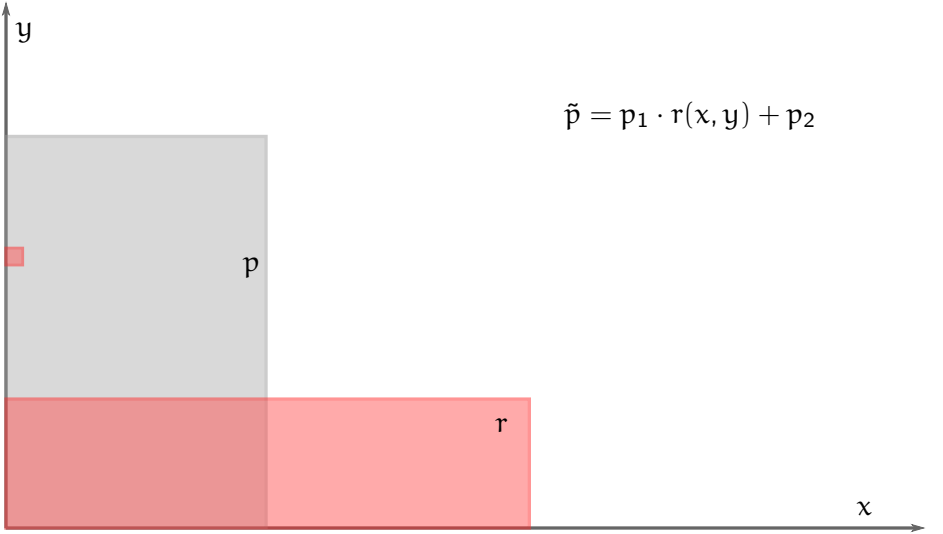
$\rightsquigarrow$ $\tilde{p}$ has x-degree $< d_x + 3n/d_y$ generically

Intermediate reshaping transformation



drawing courtesy of Grigory

Intermediate reshaping transformation



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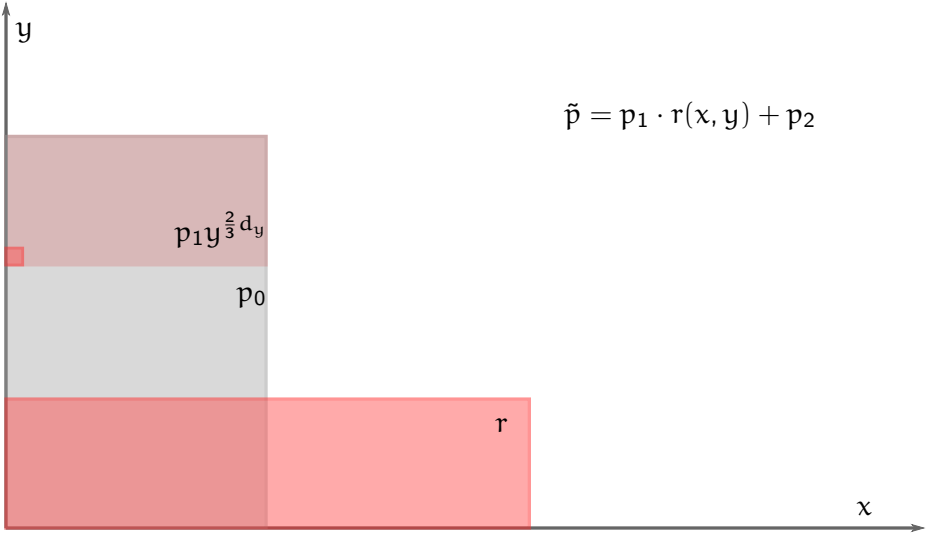

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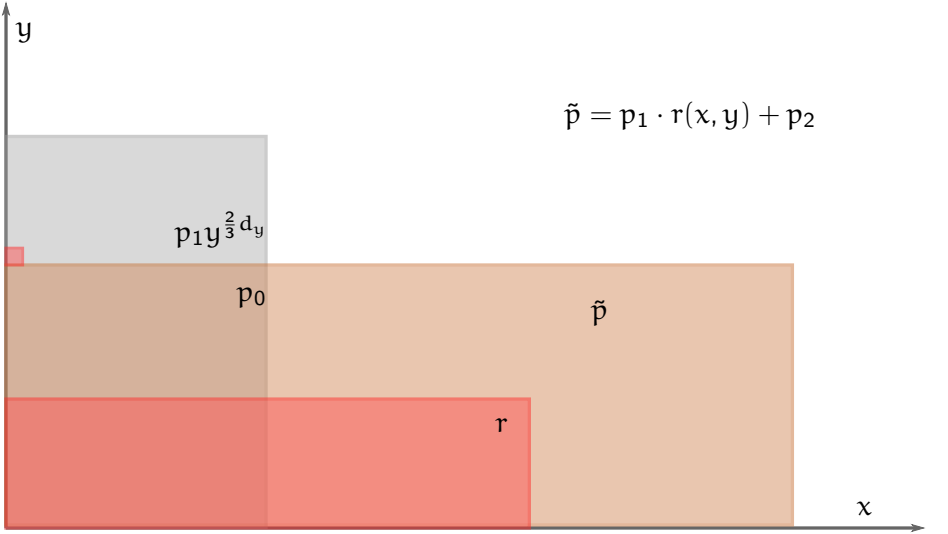

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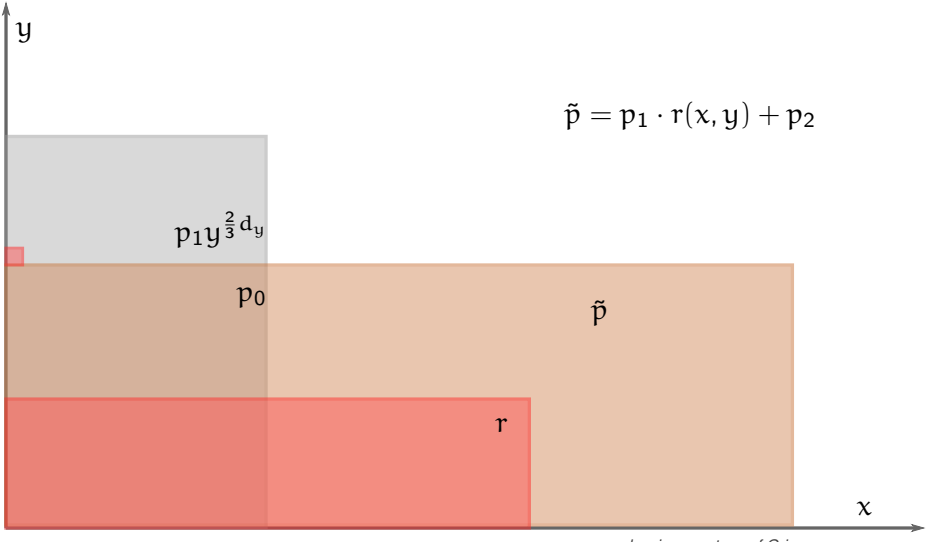


Note: $\text{size}(p) = d_x d_y \xrightarrow{\text{reshaping}} \text{size}(\tilde{p}) = (d_x + 3n/d_y)\frac{2}{3} d_y = \frac{2}{3} d_x d_y + 2n$

Iterating the reshaping: if initially $d_x d_y = n$, $\text{size}(p)$ grows to $6n$ and stabilizes

Iterated reshaping

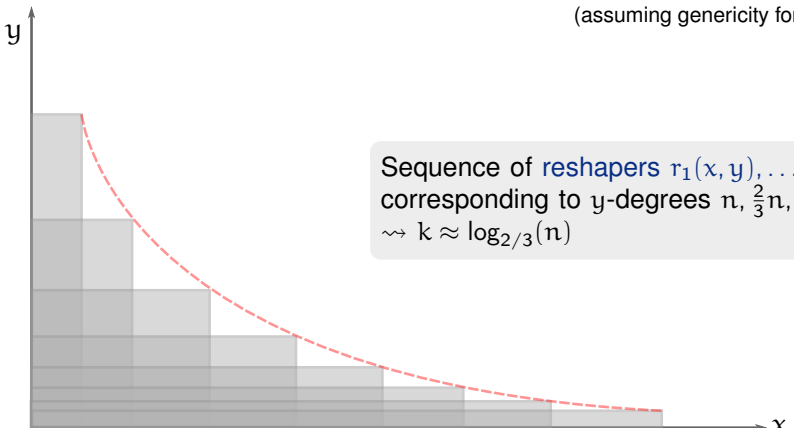
(animation courtesy of Grigory)

Iterated reshaping

Degree bounds, starting from $p(x, y)$ of degree $(0, n)$

transformation		1	2	3	$\cdots$	i	$\cdots$
x-degree	0	3	$3 + \frac{9}{2}$	$3 + \frac{9}{2} + \frac{27}{4}$	$\cdots$	$6(\frac{3}{2})^i - 6$	$\cdots$
y-degree	n	$\frac{2}{3}n$	$\frac{4}{9}n$	$\frac{8}{27}n$	$\cdots$	$(\frac{2}{3})^i n$	$\cdots$

(assuming genericity for the x-degree)



drawing courtesy of Grigory

Sequence of reshapers $r_1(x, y), \ldots, r_k(x, y)$ corresponding to y-degrees n, $\frac{2}{3}n$, $\frac{4}{9}n$, $\ldots$
$\rightsquigarrow k \approx \log_{2/3}(n)$

Algorithm and cost

Quasi-linear $\tilde{O}(n)$ for generic preinput and input $d_x d_y \in O(n)$

General: cost quasi-linear $\tilde{O}(d_x d_y + \sum_i \deg_x(r_i) \deg_y(r_i))$

Algorithm 1 RESHAPE$(p, r_1, \ldots, r_k)$

Input: bivariate polynomial $p \in \mathbb{K}[x, y]$
reshapers $r_1, \ldots, r_k$ w.r.t. $\mathcal{I} = \langle f(x), y - a(x) \rangle$
Output: univariate polynomial $\tilde{p} \in (p + \mathcal{I}) \cap \mathbb{K}[x]$ of degree $< n$

1: $\tilde{p} \leftarrow p$
2: **for** $i = 1, \ldots, k$ **do** ▷ *reshape in degree* d_i *with* $y^{\frac{2}{3}d_i} - r_i \in \mathcal{I}$
3: Write $\tilde{p} = p_1 y^{\frac{2}{3}d_i} + p_0$ where $\deg_y(p_0) < \frac{2}{3}d_i$
4: $\tilde{p} \leftarrow p_1 r_i + p_0$ ▷ *equivalent to* $\tilde{p} \leftarrow \tilde{p} - p_1 r_i \in \mathcal{I}$
5: **return** $\tilde{p} \bmod f$

Can be generalized to any ideal $\mathcal{I}$, including not zero-dimensional
. . . as long as reshapers are available!

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Bivariate modular composition

$$p(x, y) \xrightarrow{\bmod \langle f(x), y - a(x) \rangle} b(x) = p(x, a(x)) \bmod f(x)$$

The reshaping strategy applies directly
$\rightsquigarrow$ quasi-linear modular composition after precomputation on a, f

Genericity $=$ reshapers have small x-degree

- "conjectured" for any f and generic a
- observed experimentally (random a over large finite field $\mathbb{K}$)
- proved for squarefree f and generic a

  $\rightsquigarrow$ brings us to multipoint evaluation, up to a field extension

Bivariate multipoint evaluation

$$p(x, y) \xrightarrow{\bmod \langle f(x), y - a(x) \rangle} b(x) \xrightarrow{x = x_1, \ldots, x_n} p(x_1, y_1), \ldots, p(x_n, y_n)$$

The reshaping strategy applies, assuming pairwise distinct x_i's
$\rightsquigarrow$ quasi-linear bivariate evaluation after precomputation on $(x_i, y_i)_i$

- distinct coordinates $x_1, \ldots, x_n$
- generic coordinates $y_1, \ldots, y_n$

proved → small x-degree reshapers

Bivariate multipoint evaluation

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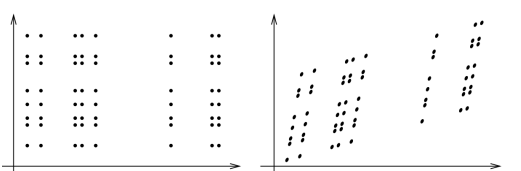
If the x_i's are not distinct, apply random point shearing [Nüsken - Ziegler 2004]

Evaluate $p(x + \theta y, y)$
at points $(x_i - \theta y_i, y_i)_i$

- distinct new x-coordinates
- new degrees $d_x + d_y, d_y$



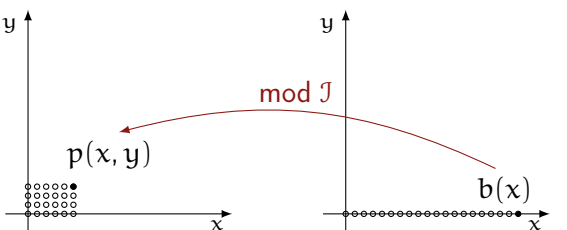
Bivariate interpolation

$$z_1, \ldots, z_n \xrightarrow[\text{interpolation}]{\text{univariate}} a(x), b(x) \xrightarrow{\bmod \langle f(x), y - a(x) \rangle} p(x, y)$$

Reshaping for $b(x) \longrightarrow p(x, y)$: precomputation = "transposed" reshapers

$\rightsquigarrow$ assuming distinct $y_1, \ldots, y_n$ and genericity in $x_1, \ldots, x_n$

quasi-linear bivariate interpolation after precomputation on $(x_i, y_i)_i$



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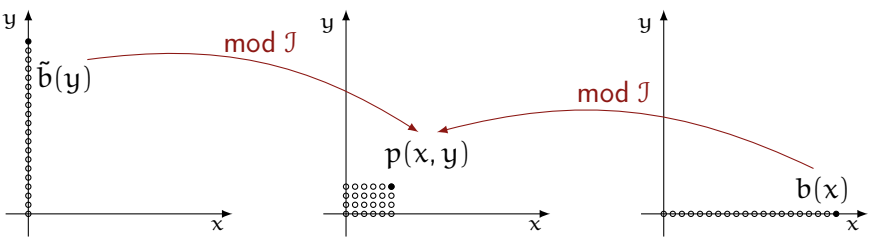
quasi-linear bivariate interpolation after precomputation on $(x_i, y_i)_i$



Bivariate interpolation

$$z_1, \dots, z_n \xrightarrow[\text{interpolation}]{\text{univariate}} a(x), b(x) \xrightarrow{\bmod \langle f(x), y - a(x) \rangle} p(x, y)$$

Reshaping for $b(x) \longrightarrow p(x, y)$: precomputation = "transposed" reshapers

$\rightsquigarrow$ assuming distinct $y_1, \dots, y_n$ and genericity in $x_1, \dots, x_n$

quasi-linear bivariate interpolation after precomputation on $(x_i, y_i)_i$

Other approach, with distinct $x_1, \dots, x_n$ and genericity in $y_1, \dots, y_n$

- Shear points ensuring distinct $\tilde{y}_1, \dots, \tilde{y}_n$
- Interpolate $\tilde{b}(y)$ such that $\tilde{b}(\tilde{y}_i) = z_i$
- Reshape $\tilde{b}(y) \longrightarrow \tilde{p}(x, y)$
- Shift $\tilde{p}$ to make it agree with non-sheared points

Bivariate interpolation

$$z_1, \dots, z_n \xrightarrow[\text{interpolation}]{\text{univariate}} a(x), b(x) \xrightarrow{\bmod \langle f(x), y - a(x) \rangle} p(x, y)$$

Reshaping for $b(x) \longrightarrow p(x, y)$: precomputation = "transposed" reshapers

$\rightsquigarrow$ assuming distinct $y_1, \dots, y_n$ and genericity in $x_1, \dots, x_n$

quasi-linear bivariate interpolation after precomputation on $(x_i, y_i)_i$

Other approach, with distinct $x_1, \dots, x_n$ and genericity in $y_1, \dots, y_n$

- Shear points ensuring distinct $\tilde{y}_1, \dots, \tilde{y}_n$
- Interpolate $\tilde{b}(y)$ such that $\tilde{b}(\tilde{y}_i) = z_i$
- Reshape $\tilde{b}(y) \longrightarrow \tilde{p}(x, y)$ with degree about $(\sqrt{n}, \sqrt{n})$
- Shift $\tilde{p}$ to make it agree with non-sheared points
- Carry on reshaping until desired y-degree

Summary and perspectives

Summary

- under genericity assumptions, and allowing precomputations
- quasi-linear algorithms for basic operations on bivariate polynomials
- main technique: reshaping degrees w.r.t. an ideal
- experiments suggest good potential in practice even in low degree

Perspectives

- Precomputation in time $\tilde{O}(n^{\frac{\omega+1}{2}})$
- Efficient implementation and study of practical impact
- Extension to several variables
- Removing genericity assumptions via randomization